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ABSTRACT

Discussion material and exercises related to pre-algebra groups are provided in this five chapter manual. Chapter 1 (mappings) focuses on restricted domains, order of operations (parentheses and exponents), rules of assignment, and computer extensions. Chapter 2 considers finite number systems, including binary operations, clock arithmetic, properties of operational systems, multiplicative systems, and subtraction/division. Other operational groups are considered in chapter 3. Additional topics include the cancellation law, non-commutative groups, and designs from finite systems. Chapter 4 focuses on integers, discussing set of integers, (Z.+) as a commutative group, subtraction, multiplication/division, order of operations, exponents, and absolute value. Transformational geometry is the major focus of chapter 5. Topics discussed in this chapter include: symmetry, plotting points, translations, translations and groups, line reflections, rotations, isometries, and dilation. Each chapter concludes with a set of review questions. In addition, a set of cumulative review questions for chapters 1-3 is included. (JN)

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PRE-ALGEBRA GROUPS
Concepts & Applications

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PRE-ALGEBRA GROUPS

Chapter 1

Mappings

CHAPTER 1 MAPPINGS

Section 1. Mappings

Have you ever sent a friend a coded message, or watched the telegraph operator in an old western movie send a message in Morse code? If so, you know that in every code each letter of the alphabet is assigned a new name. Sometimes the letters are assigned to numbers, or to other letters, or to symbols. In the case of Morse code each letter is represented by a series of dots and dashes. Whatever the process, the elements of one set are assigned to the elements of another set. Sometimes a set of people is assigned to a set of numbers, for example, each one of you is assigned a telephone number, a house number, a zip code, a student identification number and so forth.

Below are some examples where the first set is a group of four students: Ann, Jo[illegible] Mary and Linda. In their science class they all sit at the same table. The chairs are numbered 1, 2, 3 and 4. Here is an arrow diagram showing the seating arrangement.

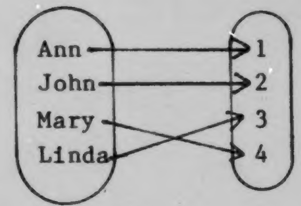


Fig. 1

In their math class they all sit in the same row.

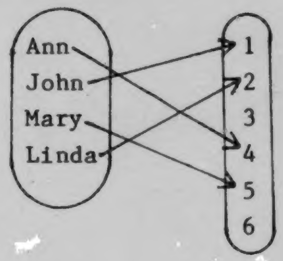


Fig. 2

Notice that there are more desks than necessary.

When the same four students ran for class office, these were the results:

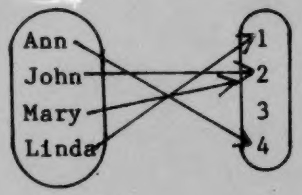


Fig. 3

No one was third, since Mary and John tied for second.

Each of these assignments has some things in common with codes. To discuss these similarities and differences, you will need to learn the definitions of some new terms. The first is the most important and the most difficult. Read it very carefully several times.

1. MAPPING A mapping assigns the elements of one set to the elements of another set in such a way that each element in the first set is assigned to exactly one element in the second set.

2. DOMAIN The domain is the name given to the first set.

3. CODOMAIN The codomain is the name given to the second set.
4. IMAGE If an element "a" in the domain is assigned to an element "b" in the codomain, then "b" is the image of "a".

5. RANGE The range is the set of all images. It may be as large as the codomain, but it is often smaller.

The terms domain, codomain, and image can be used to restate the definition of a mapping in a more concise form:

> A mapping assigns the elements of the domain to the elements of the codomain in such a way that each element in the domain has a unique image in the codomain.

Now you are ready to examine the mappings you have already met and identify the domain, codomain and range of each. Look at Figure 2. The domain is the set of four students. The codomain and range are the same, {1, 2, 3, 4}.

In figure 3 the codomain, {1, 2, 3, 4, 5, 6}, is larger than the range, {1, 2, 4, 5}. The third and sixth seats are unoccupied, that is, 3 and 6 are not used as images.

List the elements of the domain, codomain, and range for Figure. 4. This is the first mapping that you have met where an element of the codomain, in this case 2, is the image of two different domain elements, John and Mary. If this bothers you, go back and reread the definition of a mapping. It requires only that every element of the domain be assigned an image. No restrictions are made on the use of the codomain elements. In fact it is possible for all of the elements of the domain to be assigned to the same image in the codomain.

To distinguish among the types of mappings, you will need to add two new terms to your "mapping" vocabulary. If every element of the codomain serves as an image, that is, if the codomain equals the range, the mapping is called an onto mapping. Figure 2 is an example of an onto mapping. Figure 5, below, is also an onto mapping. Notice the difference between them.

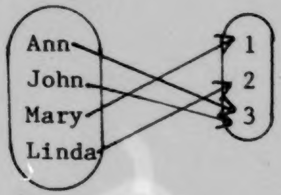


Fig. 4

If each image is the image of only one element in the domain, the mapping is called one-to-one. The mapping in Figure 2 is one-to-one, but Figure 5 is not, since 3 is the image of more than one domain element. Thus the mapping in Figure 2 is both onto and one-to-one.

Look at Figure 3. Is the mapping onto, one-to-one, or both? It is not onto because the codomain is not equal to the range. But it is a one-to-one mapping because each element of the range is the image of only one domain element.

What about Figure 4, is it an onto or a one-to-one mapping? It is not onto, the codomain and range are not equal, and it is not one-to-one because 2 is the image of two domain elements. As you can see, these two new terms separate all mappings into four categories: one-to-one mappings; onto mappings; mappings that are both onto and one-to-one; and mappings that satisfy neither criterion.

All of the assignments you have studied so far have been mappings of one type or another. However, not all assignments are mappings, as these arrow diagrams illustrate.

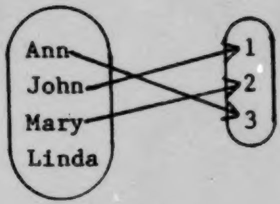


Fig. 5

Linda has no image in the codomain, so the assignment is not a mapping.

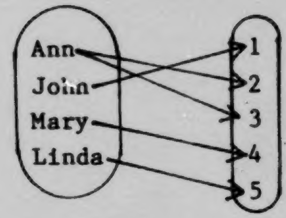


Fig. 6

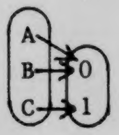
Here Ann is assigned two images, which violates that part of the definition requiring an unique image for each domain element.

You are now ready to do the exercises that follow. It will help you to distinguish between assignments that are mappings and those that are not. It will also check your understanding of the vocabulary you have learned.

EXERCISES

Study each assignment. Decide if it is a mapping or not.
Answer yes or no.

1.



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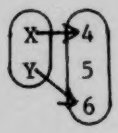

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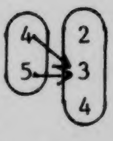

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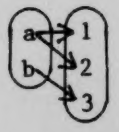

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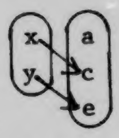

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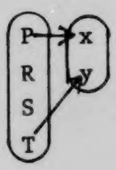

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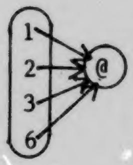

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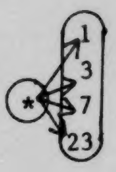
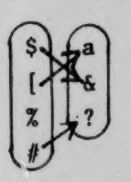


9.



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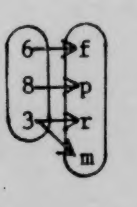

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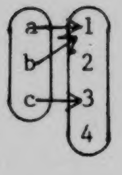
For each of the following assignments answer these questions:

- a. Is it a mapping?
- b. What is the domain?
- c. What is the codomain?
- d. What is the range?
- e. Is the mapping onto?
- f. Is the mapping one-to-one?

11.



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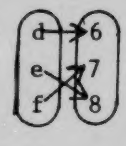

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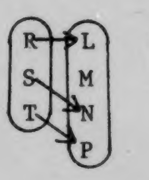

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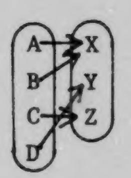

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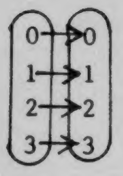


Section 2. Rules of Assignment

In the exercises, the assignments from the domain to the codomain were made randomly. Some mappings have a rule by which the assignments are made. Below is a mapping from the set of whole numbers to the set of whole numbers. If the letter W is used to denote the set of whole numbers, $W = \{0, 1, 2, 3, \ldots\}$, then this arrow diagram is a mapping from the set of whole numbers to itself.

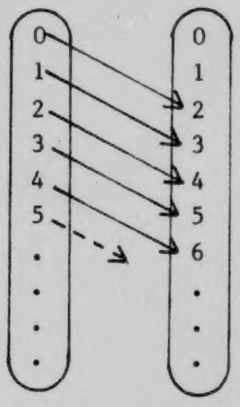


Fig. 1

Can you tell how the assignments were made? Do you see that 2 is added to each element of the domain to obtain its image in the codomain? If any domain element is represented by n, then its image is n+2, that is, $n \rightarrow n+2$, read "n maps to n+2". This is called the rule of assignment for the mapping. Here are two additional examples.

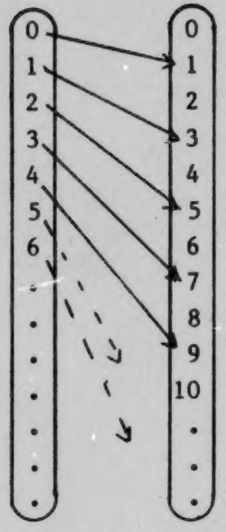


Fig. 2

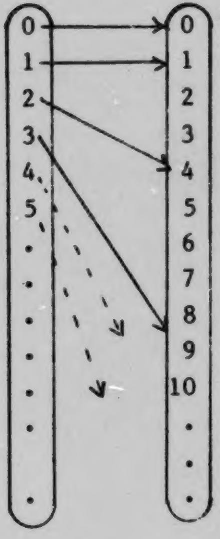
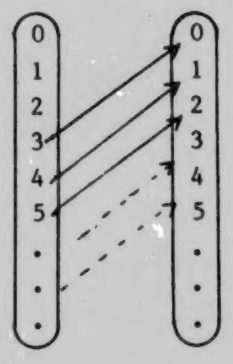


Fig. 3

Can you discover the rule of assignment for each of these two mappings? For Figure 2 the rule is $n \to 2\cdot n + 1$, while in Figure 3 the rule is $n \to n\cdot n$ or $n \to n^2$. In each of the diagrams you will notice dotted arrows coming from some of the domain elements. This is because, although each of these elements has an image, the numbers are not in the part of the codomain that is shown. Be careful to be certain that each element in the domain has an image. As an example, using the rule of assignment $n \to n - 3$ gives this partial arrow diagram.



What is the image of 0, or 1, or 2? These elements have no images. Thus the rule of assignment $n \to n-3$ does not describe a mapping from W to W.

EXERCISES

Create 10 mappings from W to W of your own. Make a partial arrow diagram, including at least the numbers 0 through 5 and their images. Be sure that the range contains no elements over 100. Write the rules of assignment in a column on the right hand side of your paper. When you come to class tear off the portion with the rules. Challenge someone in your class to write the rules for your arrow diagrams.

Section 3. Restricted Domains

To complete the exercises you probably used rules involving addition and multiplication since most rules involving subtraction and division are not mappings from W to W. Whenever these two operations are used it is necessary to restrict the domain. For example, the rule $n \rightarrow n/3$ defines a mapping to the set of whole numbers only if the domain is restricted to multiples of three, $\{0, 3, 6, 9, \ldots\}$. What is the range of this mapping? The set of whole numbers is both the codomain and the range. Therefore, this mapping is onto; it is also one-to-one.

How would you restrict the domain for this rule of assignment, $n \rightarrow n-4$? The smallest whole number that will have an image is 4. Since every whole number greater than 4 will also have an image in W, the restricted domain is $\{4, 5, 6, \ldots\}$. This mapping is onto and one-to-one.

The last example is a little more difficult. The rule is $n \rightarrow \frac{n}{2} - 3$ and involves two operations. Here each domain element is divided by 2, then 3 is subtracted from the quotient. The smallest whole number that has an image in the set W is 6. Since the image of 7 is not a whole number, it cannot be included in the domain. In fact, no odd number has an image in the set W. Thus, the restricted domain contains only even numbers greater than or equal to 6; $\{6, 8, 10, \ldots\}$. This mapping is also one-to-one and onto. The exercises that follow will give you additional examples of this type of mapping.

EXERCISES

1. Given the mapping from W to W with rule $n \rightarrow 7 \cdot n+6$, find the image of a) 7, b) 16, c) 23, d) 41, e) 0.

2. Given the mapping from W to W with rule $n \to 7\cdot(n+6)$, find the image of a) 7, b) 16, c) 23, d) 41, e) 0.

3. Given the mapping from W to W with rule $n \to 12\cdot n + 11$, find the image of a) 9, b) 17, c) 0, d) 431, e) 183.

4. Given the mapping from W to W with rule $n \to 17\cdot n + 6$, find the image of a) 9, b) 17, c) 0, d) 431, e) 196.

5. Given the mapping from W to W with rule $n \to 4\cdot n + 1$, what element of the domain will have the image
   a) 9, b) 18, c) 65, d) 81, e) 21.

6. For each of the rules below, what is the largest subset of the set of whole numbers, W, that will make the assignment a mapping?
   a) $n \to n - 5$
   b) $n \to \frac{n}{2}$
   c) $n \to \frac{n}{3} - 2$
   d) $n \to \frac{n-2}{3}$ (hint: subtract 2 from n first then divide by 3)
   e) $n \to \frac{n}{5} + 1$
   f) $n \to 3\cdot n - 4$
   g) $n \to \frac{3\cdot n - 1}{12}$

7. Which of the mappings in 6 is not onto? Why?
8. Given the rule of assignment $n \to \frac{n-9}{14}$, what is the largest subset of W that will make this a mapping?

   What is the image of
   a) 9, b) 79, c) 275, d) 331, e) 626.

   What element of the domain has as its image
   a) 6, b) 10, c) 43, d) 103, e) 0.

Section 4. Order of Operations

As you were completing the last worksheet you may have had some difficulty with those rules of assignment containing two or more operations. The difficulty was not how to do the operations but which operation to do first. There are some definite rules that tell the order in which operations are to be performed. At first you might decide to simply do each operation as you come to it, that is, the first operation first and the second operation next and so forth. Unfortunately this is not a valid rule. Study this example to see why.

You know that $1 + 6 = 6 + 1 = 7$.
You also know that $6 = 2 \cdot 3$.
Putting these two ideas together yields

$$6 + 1 = 1 + 6$$
$$2 \cdot 3 + 1 = 1 + 2 \cdot 3$$

The left side $2 \cdot 3 + 1$ is obviously equal to $6 + 1$ or 7. But the right side $1 + 2 \cdot 3$ is only equal to $1 + 6$ or 7 if the multiplication is done first. If you always did the first operation first, then $1 + 2 \cdot 3$ would equal 9. This is wrong because the answer is 7.

This leads to the first rule for the order of operations:

FOR EXPRESSIONS INVOLVING BOTH ADDITION AND MULTIPLICATION, THE MULTIPLICATION IS PERFORMED FIRST.

Ex. 1. $4 + 5 \cdot 3$
$4 + 15$
19

Ex. 2. $4 \cdot 6 + 2 + 3 \cdot 5$
$24 + 2 + 15$
41

Because of this idea these rules of assignment are equivalent.

$n \rightarrow 3 \cdot n + 4$ $\qquad$ $n \rightarrow 4 + 3 \cdot n$

Find the image of 2

$2 \rightarrow 3 \cdot 2 + 4$ $\qquad$ $2 \rightarrow 4 + 3 \cdot 2$
$2 \rightarrow 6 + 4$ $\qquad$ $2 \rightarrow 4 + 6$
$2 \rightarrow 10$ $\qquad$ $2 \rightarrow 10$

Since multiplication and division are very closely related, the rule above can probably be extended to include division. Study this example

$3 + 1 = 1 + 3$, both equal 4
$6 \div 2 + 1 = 1 + 6 \div 2$, since $6 \div 2 = 3$

The left side is obviously equal to 4. The right side is equal to 4 only if division precedes addition. The rule can now be restated: multiplication or division precedes addition.

Ex. 3 $14 \div 2 + 3$
$7 + 3$
10

Ex. 4 $3 + 14 \div 2$
$3 + 7$
10

Ex. 5 $8 \div 4 + 3 + 6 \div 2$
$2 + 3 + 3$
8

For expressions involving subtraction a similar rule applies. Consider the example below.

since $8 - 6 = 2$,
and $6 = 2 \cdot 3$,
then $8 - 2 \cdot 3 = 2$.

The only way that $8 - 2 \cdot 3 = 2$ is if the multiplication is done before the subtraction. The situation is the same for division and subtraction as shown below.

since $8 - 6 = 2$,
and $6 = 12 \div 2$,
then $8 - 12 \div 2 = 2$.

Again division must be performed before subtraction. The rule previously stated can now be extended to include all four basic operations: to simplify expressions, do multiplication and division first, followed by addition and subtraction. Some problems contain several additions and subtractions or several multiplications and divisions.

Consider this example

$$7 + 2 - 4 - 5 + 7.$$

Almost everyone would add or subtract in order from left to right yielding a result of 7. This is exactly right. Likewise for this example using only multiplication and division

$$8 \cdot 3 \div 4 \cdot 8 \div 3 \div 4$$

The operations of multiplication and division are performed in order left to right giving the result 4.

A complete rule for simplifying expressions involving the four basic operations is:

I. DO ALL MULTIPLICATIONS AND DIVISIONS IN ORDER LEFT TO RIGHT
II. DO ALL ADDITIONS AND SUBTRACTIONS IN ORDER LEFT TO RIGHT.

Study these more complex examples before attempting the exercises that follow:

Ex. 6
$8 \div 2 + 3 \cdot 4 - 5$
$4 + 12 - 5$
11

Ex. 7
$8 - 3 \cdot 2 + 4 \div 2 \cdot 3$
$8 - 6 + 6$
8

EXERCISES

1. $12 - 4 \cdot 2$
2. $18 - 8 \div 2 + 1$
3. $6 \cdot 2 - 4 \cdot 3$
4. $9 \div 3 + 2 \cdot 5$
5. $12 - 6 \div 2 + 3$
6. $8 \cdot 3 + 2 - 5 \cdot 4$
7. $30 - 4 \cdot 5 + 2 \cdot 3$
8. $5 + 8 - 6 - 3 + 2$
9. $9 - 5 - 3 + 7 - 2$
10. $8 \cdot 3 - 6 - 2 - 3$
11. $12 - 3 - 8 + 2 \cdot 6$
12. $7 \cdot 5 - 6 \cdot 3 + 2 + 4 \cdot 7$
13. $8 \div 2 \cdot 6 + 3 \cdot 4$
14. $10 \div 5 \cdot 3 + 6 \cdot 3$
15. $32 \div 8 \cdot 3 + 4 \cdot 2$
16. $40 \div 5 \cdot 3 - 40 \div 8$
17. $20 \cdot 3 - 8 \cdot 7 - 2 \cdot 2$
18. $3 \cdot 5 \cdot 4 \div 15 \cdot 2$
19. $3 + 5 + 4 - 6 \cdot 2$
20. $9 + 3 - 9 \div 3 + 9 \cdot 3$
21. $7 - 6 + 4 \cdot 5 - 2 - 4 \cdot 3$
22. $5 + 2 \cdot 5 - 6 \cdot 2 - 2 \div 2$
23. $8 - 3 \cdot 2 + 5 \cdot 6 + 16 \cdot 8$
24. $9 \cdot 6 + 5 - 7 \cdot 7 + 9 \cdot 8$
25. $12 + 4 - 6 + 8 \cdot 3 \div 12$

Section 5. Order of Operations: Parentheses

Is it ever possible to override the order of operation rules? For example, can addition ever be done before multiplication? Yes, of course it is possible, and often even necessary. Parentheses are used to override the usual order of operations. The operation inside of the parentheses is always computed first, then the rest of the expression is simplified as usual. The examples in this section are very important. Study them carefully.

Ex. 1
$8 - (3 + 2) + 4 \cdot 3$
$8 - 5 + 4 \cdot 3$
$8 - 5 + 12$
15

Ex. 2
$6 \div 3 + 8 \cdot 3 - (7 - 3)$
$6 \div 3 + 8 \cdot 3 - 4$
$2 + 24 - 4$
22

It is possible for an expression to contain two or more sets of parentheses. Simplify each set in order left to right as in the examples that follow:

Ex. 3
$(5 + 7) \div (2 \cdot 3)$
$12 \div 6$
2

Ex. 4
$12 \div (3 \cdot 2) + 9 - (5 + 2)$
$12 \div 6 + 9 - 7$
$2 + 9 - 7$
4

Sometimes one set of parentheses is needed inside another, for example $(36 - (8 - 2)) \cdot 2$. To avoid confusion the outer set of parenthesis can be replaced by square brackets. The expression will appear in the form $[36 - (8 - 2)] \cdot 2$. To simplify the expression do the innermost parentheses first, then the outer brackets:

$[36 - (8 - 2)] \cdot 2$
$[36 - 6] \cdot 2$
$30 \cdot 2$
60

Proceed very cautiously because these problems are tricky. Do not attempt many short cuts.

Ex. 5
$23 - [2 \cdot (3 + 4) + 3]$
$23 - [2 \cdot 7 + 3]$
$23 - 17$
6

Ex. 6
$8 \div [12 \div (4 + 2)]$
$8 \div [12 \div 6]$
$8 \div 2$
4

Problems involving division may be written two ways; $6 \div 2$ or $6/2$. Example 6 above might also be written:

$$\frac{8}{12 \div (4 + 2)}$$

Notice that the brackets around the denominator have been eliminated. The fraction bar has taken their place. A general rule for simplifying expressions of this type is: simplify the numerator, simplify the denominator, and then divide the numerator by the denominator.

Ex. 7

$$\frac{4 + 2\cdot 3 + 5 - 3}{2\cdot(5 - 2)}$$

$$\frac{4 + 6 + 5 - 3}{2\cdot 3}$$

$$\frac{12}{6}$$

$$2$$

Ex. 8

$$\frac{9 - (3 + 2) + 12\cdot 3\div 6}{14 - 2\cdot 4 - (4\div 4)}$$

$$\frac{9 - 5 + 12\cdot 3\div 6}{14 - 2\cdot 4 - 1}$$

$$\frac{9 - 5 + 6}{14 - 8 - 1}$$

$$\frac{10}{5}$$

$$2$$

To Do the Exercises Below Follow These Rules:

1. CLEAR PARENTHESES AND BRACKETS USING THE RULES THAT FOLLOW
2. DO MULTIPLICATION AND DIVISION IN ORDER FROM LEFT TO RIGHT
3. DO ADDITION AND SUBTRACTION IN ORDER FROM LEFT TO RIGHT

EXERCISES

1. $36 - 8\div 4\cdot 2 + 6$
2. $36 - (8\div 4\cdot 2 + 6)$
3. $36 - 8\div 4\cdot(2 + 6)$
4. $36 - 8\div(4\cdot 2) + 6$
5. $(36 - 8)\div 4\cdot(2 + 4)$
6. $36 - [8\div(4\cdot 2) + 6]$
7. $(36 - 8)\div(4\cdot 2 + 6)$
8. $12 - 4 + 3\cdot 2 + 4$
9. $(12 - 4 + 3)\cdot 2 + 4$
10. $[12 - (4 + 3)]\cdot(2 + 4)$
11. $12\cdot 4 + 3\cdot(2 + 4)$
12. $144\div 12 + 4\cdot 8\div 2$
13. $144\div(12 + 4)\cdot(8\div 2)$
14. $144\div 12 + 4\cdot(8\div 2)$
15. $144\div(12 + 4)\cdot 8\div 2$
16. $180\div 12\cdot 3 + 2\cdot 6 - 3$
17. $180\div 12\cdot 3 + 2\cdot(6 - 3)$
18. $180\div 12\cdot(3 + 2)\cdot(6 - 3)$
19. $180\div 12\cdot(3 + 2\cdot 6) - 3$
20. $180\div(12\cdot 3) + 2\cdot 6 - 3$
21. $180\div 12\cdot(3 + 2\cdot 6 - 3)$
22. $180\div 12\cdot(3 + 2)\cdot 6 - 3$
23. $180\div[12\cdot(3 + 2)]\cdot 6 - 3$
24. $180\div(12\cdot 3) + 2\cdot(6 - 3)$
25. $[180\div(12\cdot 3) + 2]\cdot(6 - 3)$

Supply parentheses and brackets to make each of the following statements true. Some problems require no parentheses at all. Always use as few parentheses as possible.

26. $18 - 6 \div 3 \cdot 2 + 1 = 9$

27. $18 - 6 \div 3 \cdot 2 + 1 = 15$

28. $18 - 6 \div 3 \cdot 2 + 1 = 48$

29. $18 - 6 \div 3 \cdot 2 + 1 = 3$

30. $18 - 6 \div 3 \cdot 2 + 1 = 16$

31. $18 - 6 \div 3 \cdot 2 + 1 = 12$

32. $18 - 6 \div 3 \cdot 2 + 1 = 18$

33. $18 - 6 \div 3 \cdot 2 + 1 = 33$

34. $36 - 8 \div 4 \cdot 2 + 6 = 38$

35. $36 - 8 \div 4 \cdot 2 + 6 = 26$

36. $36 - 8 \div 4 \cdot 2 + 6 = 20$

37. $36 - 8 \div 4 \cdot 2 + 6 = 41$

38. $36 - 8 \div 4 \cdot 2 + 6 = 29$

39. $36 - 8 \div 4 \cdot 2 + 6 = 2$

40. $36 - 8 \div 4 \cdot 2 + 6 = 56$

41. $36 - 8 \div 4 \cdot 2 + 6 = 74$

Simplify each expression.

42. $\frac{3\cdot 4 + 2}{9 - 2}$

43. $\frac{7 - (3 + 2) + 6}{8\div 2}$

44. $\frac{36\div 2\div 3 + 3\cdot(2 + 5)}{12\div 6\cdot 3 - (7 - 20\div 5)}$

45. $\frac{5\cdot(3 + 7) - 2}{6 + 7\cdot 6}$

46. $\frac{8\div 2 - 4\div 2 + 7}{6\div 2 - 2}$

47. $\frac{9\cdot 7 - 3\cdot 5 + 2}{20\div(8 - 3) + 3\cdot 2}$

48. $\frac{49\div(3 + 4) + 3\cdot(5 + 3\cdot 6)}{3 + 4\cdot(6 - 2)}$

49. $\frac{(7 - 3)\cdot 5 + 2\cdot 8 - 6}{(6 - 2)\div 2 + 15\div 5}$

50. $\frac{7 + 75\div 5 - 6 - 2 - 4}{5 - [(8 + 7)\cdot 3 - 6\cdot 7]}$

51. $\frac{12\cdot(4 + 6\cdot 3) - 8\cdot 9}{36\div(2\cdot 3) + 4\cdot 3\div 2}$

52. $\frac{6\cdot 7\cdot 8 - 5\cdot 4\cdot 3}{91\div 7 + 45\div 9\cdot 2}$

53. $\frac{2\cdot 4 + 6}{14\div 2} + \frac{2\cdot(4 + 6)}{10\div 2 - 1}$

54. $\frac{3\cdot(4 + 2\cdot 5)}{7} - \frac{5\cdot 6 - 8\cdot 3}{2\cdot 4 - 5}$

55. $\frac{15\cdot 3 - (4 + 2)}{7 - 3\cdot 2} - [3\cdot(2\cdot 2 - 1)]$

Section 6. Order of Operations: Exponents

Suppose you need to multiply 2 by itself ten times. You could write

$$2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2$$

but this would be very long and tedious. To make this problem much simpler, exponents are used. The expression above is then written as

$$2^{10}$$

The exponent is 10. It tells how many times 2 is used as a factor. Two is called the base. The expression is read "two to the tenth power". Here is another example:

$$5^4 = 5\cdot5\cdot5\cdot5 = 625$$

The expression is read "five to the fourth power". Five is the base and four is the exponent.

In general, a^n, read a to the n^{th} power, means to use the base, a, as a factor, n times.

Special names are given to the exponents 2 and 3. The expression 5^2 is usually read "five squared" rather than "five to the second power". The expression 7^3 is read "seven cubed" rather than "7 to the third power". This is done because of their relationship to the area of a square and volume of a cube, as you will see later.

Exponents are often used in expressions like the ones in the previous exercises. Therefore it is necessary to decide when to evaluate them. The answer is very simple: Exponents are a shorthand for multiplication and are evaluated first. Study the following example to see why the exponent is evaluated even before the other multiplication is carried out.

$8 \cdot 3 = 3 \cdot 8$ (both are equal to 24)
$2^3 \cdot 3 = 3 \cdot 2^3$ (since $8 = 2 \cdot 2 \cdot 2 = 2^3$)

Most of you would agree that $2^3 \cdot 3 = 24$ but some would probably say that $3 \cdot 2^3 = 6^3 = 6 \cdot 6 \cdot 6 = 216$, if you multiplied the $3 \cdot 2$ first. You already know that the correct answer is not 216 but 24. And 24 can result only if the exponent is evaluated first, that is, $3 \cdot 2^3 = 3 \cdot (2 \cdot 2 \cdot 2) = 3 \cdot 8 = 24$.

Ex. 1
$4 \cdot 3^2 + 2^2 \cdot 5$
$4 \cdot 9 + 4 \cdot 5$
$36 + 20$
56

Ex. 2
$5 \cdot 2^4 - 3^2 \cdot 2^3$
$5 \cdot 16 - 9 \cdot 8$
$80 - 72$
8

Occasionally parentheses are involved. As usual, they must be evaluated first. Study these examples:

Ex. 3
$(3 + 2)^2$
5^2
25

Ex. 4
$(2 \cdot 3)^2$
6^2
36

Ex. 5
$(3 + 2 \cdot 5)^2$
$(3 + 10)^2$
13^2
169

Before considering some final examples, it is appropriate to summarize all the order of operation rules.

EVALUATE ALL EXPRESSION IN THIS ORDER:
I. PARENTHESES ARE CLEARED FIRST ACCORDING TO THE RULES THAT FOLLOW
II. EXPONENTS
III. MULTIPLICATION AND DIVISION IN ORDER, LEFT TO RIGHT
IV. ADDITION AND SUBTRACTION IN ORDER, LEFT TO RIGHT

Some very complex expressions can be evaluated using these rules. By writing out several intermediate steps and not doing too many mental calculations, you will be able to arrive at the correct result. Take your time, study the problem carefully to determine the proper order of operations. These last examples should help to prepare you for the exercises that follow.

Ex. 6 $4 \cdot 5 + 3^2 - 4 - (3 \cdot 2^2)$
$4 \cdot 5 + 3^2 - 4 - (3 \cdot 4)$
$4 \cdot 5 + 9 - 4 - 3 \cdot 4$
$20 + 9 - 4 - 12$
13

Ex. 7 $(5 - 2)^2 + 4 \cdot 2^3 - 3$
$3^2 + 4 \cdot 2^3 - 3$
$9 + 4 \cdot 8 - 3$
$9 + 32 - 3$
38

Ex. 8 $2^3 \cdot (3 + 4) - (3^2 - 5)^2$

$2^3 \cdot 7 - (9 - 5)^2$

$2^3 \cdot 7 - 4^2$

$8 \cdot 7 - 16$

$56 - 16$
40

Ex. 9 $\dfrac{7^2 - 2^4 + 2 \cdot 3^2}{2^2 \cdot 3^2 \div 2 - 1}$
$\dfrac{49 - 16 + 2 \cdot 9}{4 \cdot 9 \div 2 - 1}$
$\dfrac{49 - 16 + 18}{4 \cdot 9 \div 2 - 1}$
$\dfrac{51}{17}$
3

EXERCISES

Use exponents to write each of the following expressions:

1. $5 \cdot 5 \cdot 5$
2. $3 \cdot 3 \cdot 3 \cdot 3$
3. $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$
4. $6 \cdot 6$
5. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 3$

Simplify each expression

6. 4^3
7. 2^6
8. 5^3
9. 3^5
10. $2^3 \cdot 3^2$
11. $2^3 + 3^2$
12. $3 \cdot 2^2$
13. $(3 \cdot 2)^2$
14. $5^2 + 2^2$
15. $(5 + 2)^2$
16. $2^3 \cdot 3^3$
17. $(2 \cdot 3)^3$
18. $2^3 + 3^3$
19. $(2 + 3)^3$
20. $7^3 + 4^2$

Determine whether the following statements are true or false. If a statement is false, replace the equal sign with <, "less than", or >, "greater than", to make a true statement.

Ex.: $(2 + 3)^2 = 2^2 + 3^2$

$5^2 = 4 + 9$

$25 = 13$

FALSE

$(2 + 3)^2 > 2^2 + 3^2$

21. $15^2 = 3 \cdot 5^2$
22. $2^2 + 5 = 3^2$
23. $(5 \cdot 7)^2 = 5^2 \cdot 7^2$
24. $(5 + 7)^2 = 5^2 + 7^2$
25. $(5 + 7)^2 = 12^2$
26. $(7 - 3)^2 = 7^2 - 3^2$
27. $(8 \div 2)^3 = 8^3 \div 2^3$
28. $5^2 + 2^2 = (5 \cdot 2)^2$

Simplify each expression:

29. $3^4 + 2^2$
30. $3 \cdot 2^4$
31. $7^2 + 2 \cdot 3^3$
32. $2 \cdot 3^3 - 7^2$
33. $(2 \cdot 5)^3 \div (2 \cdot 5^2)$
34. $(5 - 3)^3 \cdot 3$
35. $6 \cdot 2^2 + 5 \cdot (3^2 - 1)$
36. $48 \div 6 \cdot 2^3 + 4 \cdot 3^2$
37. $48 \div (6 \cdot 2^3) + 4 \cdot 3^2$
38. $48 \div 6 \cdot (2^3 + 4) \cdot 3^2$
39. $48 \div 6 \cdot 2^3 + (4 \cdot 3)^2$
40. $(48 \div 6 \cdot 2^3 + 4 \cdot 3)^2$
41. $(4 + 3)^2 \cdot 2 + 4^2 - 1$
42. $(3 + 2)^3 + (5 - 2)^4$
43. $(3 + 2)^3 + 5 - 2^4$
44. $(3 - 2)^3 + 7 - 2^2$
45. $8^2 - 8 + 3^2 - 3$
46. $8^2 - (8 + 3^2) - 3$
47. $(5 \cdot 4 + 2)^2 + (3 - 1)^2$
48. $5 \cdot (4 + 2)^2 + 3^2 - 1^4$
49. $[2 \cdot (3 + 1)^2]^2$
50. $(3^2 + 2)^2 - (2^2 + 3)^2$
51. $\frac{(3 + 2)^3 - 2^3}{3^2}$
52. $\frac{(7 - 2^2)^2}{2^2 - 1}$
53. $\frac{3^2 \cdot (12 - 2^3)}{2 \cdot 3^2}$
54. $\frac{2^5 - 2^4}{2^3 - 2^2}$
55. $\frac{(2^3 - 2^2)^2}{5 \cdot (2^2 - 1) + 1}$

Section 7. Rules of Assignment

In the previous exercises you simplified expressions by applying rules which determine the order in which operations are performed. These ideas can be applied to any rule of assignment. Given any domain element it is possible to find its image no matter how complex the rule of assignment. For example, using this rule, find the image of 5.

$$\begin{aligned}
n &\rightarrow n\cdot(3 + n) - 2\cdot n\\
5 &\rightarrow 5\cdot(3 + 5) - 2\cdot 5\\
5 &\rightarrow 5\cdot 8 - 2\cdot 5\\
5 &\rightarrow 40 - 10\\
5 &\rightarrow 30
\end{aligned}$$

This example involves exponents. Again find the image of 5.

$$\begin{aligned}
n &\rightarrow 3\cdot n^{2} - 4\cdot n + 2\\
5 &\rightarrow 3\cdot 5^{2} - 4\cdot 5 + 2\\
5 &\rightarrow 3\cdot 25 - 4\cdot 5 + 2\\
5 &\rightarrow 75 - 20 + 2\\
5 &\rightarrow 57
\end{aligned}$$

But beware, the more complex the rule of assignment, the more difficult it is to decide if the rule describes a mapping from W to W. Frequently you will have to find the images of several domain elements to be certain. It is also more difficult to determine the restricted domain. Study this rule of assignment carefully

$$n \rightarrow 3\cdot n^{2} + 2\cdot n - 5.$$

At first it may seem that this rule describes a mapping from W to W. Find the images of 1 and 2.

$$\begin{aligned}
n &\rightarrow 3\cdot n^{2} + 2\cdot n - 5 & \qquad n &\rightarrow 3\cdot n^{2} + 2\cdot n - 5\\
1 &\rightarrow 3\cdot 1^{2} + 2\cdot 1 - 5 & \qquad 2 &\rightarrow 3\cdot 2^{2} + 2\cdot 2 - 5\\
1 &\rightarrow 3\cdot 1 + 2\cdot 1 - 5 & \qquad 2 &\rightarrow 3\cdot 4 + 2\cdot 2 - 5\\
1 &\rightarrow 3 + 2 - 5 & \qquad 2 &\rightarrow 12 + 4 - 5\\
1 &\rightarrow 0 & \qquad 2 &\rightarrow 11
\end{aligned}$$

Larger values of n would seem to have larger and larger images. But what is the image of 0?

$$n \to 3\cdot n^2 + 2\cdot n - 5$$
$$0 \to 3\cdot 0^2 + 2\cdot 0 - 5$$
$$0 \to 3\cdot 0 + 2\cdot 0 - 5$$
$$0 \to 0 + 0 - 5$$
$$0 \to 0 - 5$$

Since 0 - 5 is not a whole number this rule of assignment does not describe a mapping from the set of whole numbers, W, to the set of whole numbers. It is necessary to restrict the domain by excluding 0. Then the rule will describe a mapping from N (the natural numbers) to the set W.

The following exercises require more careful thinking than some others. Take your time!

EXERCISES

1. Find the image of each domain element for the rule of assignment,
   $n \to 2\cdot n^2 + 3\cdot n + 5$
   a) 0 b) 1 c) 3 d) 6 e) 9

2. Find the image of each domain element for the rule of assignment,
   $n \to 3\cdot n^2 - 4\cdot n + 6$
   a) 0 b) 3 c) 7 d) 8 e) 9

3. Find the image of each domain element for the rule of assignment,
   $n \to 2\cdot n^2 - 5\cdot n + 1$
   a) 0 b) 1 c) 2 d) 4 e) 7

4. Find the image of each domain element for the rule of assignment,
   $n \to \frac{3\cdot(n^2 + 2\cdot n)}{n + 2}$

   a) 0 b) 1 c) 2 d) 5 e) 7

5. Given the rule of assignment $n \to 3\cdot n^2 + 2\cdot n$, what element of the domain, W, has as its image:
   a) 0 b) 16 c) 85 d) 5 e) 208

6. Given the rule of assignment $n \to n^2 + 4\cdot n + 3$, what element of the domain W has as its image:

   a) 24 b) 8 c) 48 d) 3 e) 63

7. Given the rule of assignment $n \to n^2 + 10 - 6\cdot n$, find the image of each whole number less than or equal to 10. Does this rule describe a mapping from the set W to the set W? Is the mapping onto? Is the mapping one-to-one?

8. Determine if each rule of assignment describes a mapping from the set W to the set W. If the rule does not describe a mapping, give at least one domain element that has no image in the set W.

   a) $n \to n^2 - 4$
   b) $n \to 2\cdot(n + 3) + n^2$
   c) $n \to 5\cdot n + 3\cdot n^2 - 2$
   d) $n \to \frac{n}{2} + 3$
   e) $n \to 3\cdot n^2 - n + 2$
   f) $n \to \frac{2\cdot n^2 + 4}{2}$
   g) $n \to \frac{5\cdot n - 4}{3}$
   h) $n \to 5\cdot n\cdot(n + 4)$
   i) $n \to n^2 - 5\cdot n - 1$
   j) $n \to 2\cdot n^2 + 4 - 3\cdot n$
   k) $n \to (n + 2)\cdot(n + 4)$
   l) $n \to \frac{n^3 + 4\cdot n^2}{n + 4}$

9. List the restricted domain for each rule above that is not a mapping.

Section 8. Computer Extension

You can use a computer to simplify expressions, too. It has been programmed to use the same rules for the order of operations that you have learned. There are some differences in the symbols that you and the computer use. The list below shows how each operation is represented in BASIC and some other computer languages.

symbol	meaning	example
+	addition	3 + 5
-	subtraction	3 - 8
*	multiplication	3 * 9
/	division	12/4
^	exponentiation	2^3

NOTE: use Shift N to get this last symbol on some computer terminals.

In typing BASIC instructions spaces are ignored:

526*   31 and 526*31 are the same.

If you wanted to evaluate the expression,

3·(6 + 4)

you need to make one symbol change. The multiplication dot · becomes *. You would type

3 * (6 + 4)

Here are some examples of expressions and their BASIC equivalents

Ex. 1 20   4•3 - 2•(6 - 2)
20/4*3-2*(6-2)

Ex. 2 20÷ (2•3 - 2) + 3•(4÷2)
20/(2*3-2) + 3*(4/2)

Expressions involving exponents look quite different because the whole expression is typed on one line. There are no raised exponents. Press shift and N to get the symbol ^. For example

3^4 = 3^4 and $3 \cdot 5^2$ = 3*5^2.

If 3*5^2 seems confusing, then add a set of parentheses: 3*(5^2).
This example also involves exponents

Ex. 3 $2 \cdot 5^2 - 4 \cdot 3 + 2^3$
= 2*5^2 - 4*3 + 2^3 or 2*(5^2)-4*3+(2^3)

Expressions written in fractional form must be changed to fit onto one line. To do this, type the numerator in parentheses, then / for the fraction bar, and then the denominator, also in parentheses.

Ex. 4 $\frac{7+5}{4}$ = (7 + 5)/4

Ex. 5 $\frac{6 \cdot 3 + 2}{4 + 3 \cdot 2}$ = (6*3 + 2)/(4 + 3•2)

The computer does not use brackets in arithmetic expressions, so replace any brackets with parentheses. Some expressions will then have multiple sets of parentheses. It is usually necessary to count the left and right parentheses to be certain that there are the same numbers of each.

Ex. 6 $$\frac{3\cdot(2 + 8)}{2^{3} + 7} = \frac{3*(2 + 8)}{2\hat{}3 + 7} = (3*(2+8))/(2\hat{}3 + 7)$$

Ex. 7 $$\frac{(3 + 2)^{3} - 2^{3}}{3^{2}} = \frac{(3 +2)\hat{}3 - 2\hat{}3}{3\hat{}2} = ((3+2)\hat{}3-2\hat{}3)/(3\hat{}2)$$

You are now ready to proceed to the computer to try it. If someone has been working before you, then begin by typing NEW and push RETURN. This will clear the other persons's work so it will not be confused with yours.

Now you are ready to write your first program. Type

```
10 PRINT (3+2)*5
```

Push RETURN. Type RUN and push RETURN again. There should be the answer to your calculation, 25. But, if you did not get 25, try again. Start at the beginning. You should check to be sure to:

1)Type RETURN after EVERY line.
2)Type NEW before starting a new program.
3)Type 10 PRINT and then your expression.
4)Type RUN. This actually starts the computer working on your answer.

What you see on the screen should look exactly like this:

```
NEW
OK?
10 PRINT (3+2)*5
RUN
25
OK?
```

Try another expression:

$$3 + 5^{2} - (4 + 3\cdot 7) = 3+5\hat{}2-(4+3*7)$$

Type this exactly:

```
NEW (return)
10 PRINT 3+5^2-(4+3*7) (return)
RUN (return)
```

The computer should print

```
3
OK?
```

The OK? tells you that the program is finished running, and it is asking for any additional instructions.

At this point you should have some questions about what you typed. For example, what is the purpose of the 10, is the word PRINT necessary, and can the computer do several expressions at one time? Here are the answers to these question: Ten is the line number. Every line of a program is numbered. Usually it is best to number the lines by 10's, that is, 10, 20, 30, 40, ... Then, if you decide to add another line between two lines already typed, you can do so. The order in which instructions(statements) are executed depends on their line numbers and not on the order in which you typed them. The instruction with the smallest line number is executed first, even if it was not typed first. If two instructions have the same line number, then the first statement will be discarded and only the second statement is kept as part of the program. The computer always sorts the statements into numerical order before executing the program.

The use of the print statement will be discussed in detail later, but for now you will need to include at least one print statement in each program because otherwise the computer will not tell you the results of your program. A print statement always has a line number. After PRINT comes the expression you want to evaluate.

The computer is capable of simplifying as many expressions as you would like at one time. Each expression is preceeded by a print statement. For example, simplify:

$$3+6\cdot5-2^4 \qquad 2\cdot7^2-3 \qquad \frac{3^3+5-2}{2^3-6\div2}$$

Rewrite each expression:

```
3+6*5-2^4        2*7^2-3        (3^3+5-2)/(2^3-6/2)
```

Now the program would be:

```
NEW
10 PRINT 3+6*5-2^4
20 PRINT 2*7^2-3
30 PRINT (3^3+5-2)/(2^3-6/2)
RUN
```

The computer will print:

```
17
95
6
OK?
```

What about errors, how can they be corrected? There are several ways to correct mistakes. At this time, whenever you make a mistake, press return and retype the line using the same line number.

EXERCISES

Rewrite each expression using computer symbols

1. $(5+7)\cdot 3$
2. $2^3 + 6\cdot 5$
3. $(7\cdot 6 + 3)\div 5$
4. $12 \div 6\cdot 2 + 5$
5. $12 \div (6\cdot 2) + 5$
6. $3^2 - 5$
7. $2\cdot 3^4$
8. $(2\cdot 3)^4$
9. $7 + 2^3 - 15 \div 5$
10. $3^2\cdot 2^3$
11. $6+3\cdot 2^4 - 2^3\cdot 5$
12. $\frac{9 + 3\cdot 2}{5}$
13. $\frac{3^2+6\cdot 2}{2^3-1}$
14. $\frac{15 - (3 + 2^2)}{2^2}$
15. $\frac{(4+3)^2 + 1}{2\cdot 5}$
16. $[2\cdot (3 + 2)^2]^2$

Find and correct the error in each program:

17.
```
NEW
10 PRINT 6 + 3•4 ÷ 2
RUN
```

18.
```
NEW
PRINT 3*6/2
RUN
```

19.
```
NEW
10PRINT 4+6^3
20 RUN
```

20.
```
NEW
20 PRINT (6+3)*5
30 RUN
```

21.
```
NEW
10 PRYNT 3+8+6
RUN
```

22.
```
NEW
100 PRNT (3+4)*6**5
RUN
```

Write a program to evaluate each expression:

23. $(5-3)^3 + 2$

24. $6\cdot2^2 + 5\cdot(3^2 - 1)$

25. $\dfrac{(9+5)\cdot4}{2^3 - 1}$

26. $6^2 \div (3\cdot2^2)$

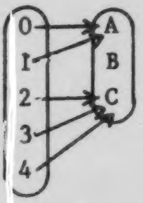
27. $8^2 - (8+3^2) - 3$

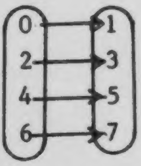
28. $\dfrac{2^5 - 2^4}{2^3 - 2^2}$

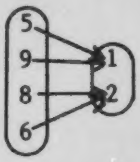
REVIEW MAPPINGS & ORDER OF OPERATIONS

1. Define these terms: mapping, domain, codomain, range, image, rule of assignment, onto mapping, one-to-one mapping.

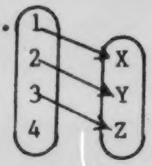
2. Answer these questions about each arrow diagram
   1. Is it a mapping?
   2. List the domain
   3. List the codomain
   4. List the range
   5. Is the mapping onto?
   6. Is the mapping one-to-one?

A.


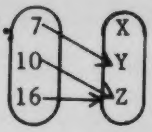
B.


C.


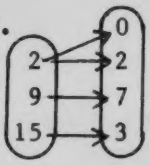
D.



E.

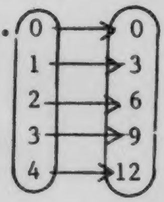


F.

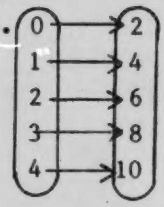


3. Write the rule of assignment for each mapping, given the partial arrow diagram shown below.

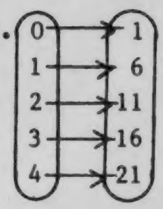
A.



B.



C.



4. Find the image of 0, 3, 6, 7, for each rule of assignment given below.

A. $n \to 3\cdot n+2$ B. $n \to 4\cdot(n^2+5)$ C. $n \to n^2+2\cdot(n+4)$

5. Find the domain element that has as its image, 9, 13, and 25, for each rule of assignment given below.

A. $n \to 2\cdot n+1$ B. $n \to 4\cdot n+5$ C. $n \to 2\cdot n+3$

6. What is the largest subset of the set, W, for which the rule describes a mapping to the set of whole numbers W?

A. $n \to 2\cdot n-5$ B. $n \to \frac{n-5}{2}$ C. $n \to \frac{n-3}{5}$ D. $n \to \frac{4\cdot n-5}{3}$

E. $n \to 3\cdot n^2-2\cdot n$ F. $n \to 2\cdot n^2-5\cdot n-1$ G. $n \to \frac{2\cdot n+3}{4}$

7. Simplify each expression.

A) $40-(8+2) \div 2 + 6 \cdot 3$

B) $40-8+2 \div 2 + 6 \cdot 3$

C) $60 \div (4+6) \div 3 \cdot 2$

D) $60 \div 4 + 6 \div 3 \cdot 2$

E) $60 \div 4 + 6 \div (3 \cdot 2)$

F) $60 \div [4 + 6 \div (3 \cdot 2)]$

G) $60 \div (4 + 6) \div (3 \cdot 2)$

H) $3 \cdot 2^4 - 3 \cdot (7-2)$

I) $5^2 - 3^2 + 2 \cdot (2^3 - 1)$

J) $7^2 \cdot 3 \cdot (2^3 - 6)$

K) $\dfrac{3^4 - 3^2 \cdot 5}{24 - 3 \cdot (3^2 - 7)}$

L) $5^3 \cdot 2 \div 10 + 3^2 \cdot (2^3 - 5)$

M) $\dfrac{12^2 \div 2^4 \cdot 3^2 + 2^2}{2^4 + 1}$

N) $\dfrac{19 + 3^2 - 2^2 + 14 - 7 \cdot 3}{27 \div 3 \cdot 2 - (2^3 - 7)}$

8. Supply the necessary grouping symbols.

A) 120 ÷ 4•3•2 + 6•2 = 17
B) 120 ÷ 4•3•2 + 6•2 = 192
C) 120 ÷ 4•3•2 + 6•2 = 32
D) 120 ÷ 4•3•2 + 6•2 = 8
E) 120 ÷ 4•3•2 + 6•2 = 2

Answers

1. See Text. 2. A. 1) yes 2) {0, 1, 2, 3, 4} 3) {A, B, C} 4) {A,C} 5) no 6)no B. 1) yes 2) {0, 2, 4, 6} 3) {1, 3, 5, 7} 4) {1, 3, 5, 7} 5) yes 6) yes C. 1) yes 2) {5,9,8,6} 3) {1,2} 4) {1,2} 5) yes 6) no D. 1) no E. 1) yes 2) {7,10,16} 3) {x,y,z} 4) {y,z} 5) no 6) no F. 1) no 3 A. n → 3•n B. n →2•(n+1) or 2•n+2 5) n → 5•n+1 4A. 2,11,20,23 B. 20,56,164,216 C. 8,23,56,71 5A. 4,6,12 B. 1,2, C. 3,5,11 6A. {3,4,5...} B.{5,7,9...} C. {15,20,25...} D. {2,5,8,11...} E)W F){3,4,5,...} G. none 7 A) 53 B) 51 C) 4 D) 19 E) 16 F) 12 G) 1 H)33 I) 30 J) 294 K) 2 L) 52 M) 5 N) 1 8A. 120 ÷ (4•3•2)+6•2 = 17 B.none C. 120 ÷ (4•3)•2+6•2=32 D. 120 ÷ (4•3•2+6)•2=8 E. 120 ÷ [(4•3•2+6)•2]=2

PRE-ALGEBRA GROUPS

Chapter 2

Finite Number Systems

Section 1. Binary Operations

Until now you have been using a rule of assignment to map individual elements of the set of whole numbers,W, to the set of whole numbers. Addition is also a mapping from the set W to the set W which combines pairs of elements in W to obtain an image in W. In this case, the domain consists of ordered pairs of whole numbers, for example; (4,1), (3,8), (5,2). The image of each pair is its sum.

Below is a partial arrow diagram for the addition of whole numbers.

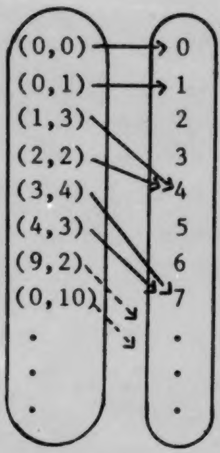


Fig. 1

Since each of the whole numbers in the codomain will eventually be the image of an ordered pair from the domain, the mapping is onto. But it is not one-to-one. For example, 4 is the image of several ordered pairs; (0,4), (4,0), (1,3), (3,1), (2,2). Although they have the same image, the ordered pairs (1,3) and (3,1) are not the same. You will understand the need for this distinction very soon.

Multiplication is a similar mapping where every ordered pair in W is assigned to its product. Subtraction in the set of whole numbers is a different situation altogether. Study the partial arrow diagram in Fig. 2. See if you recognize the difficulty.

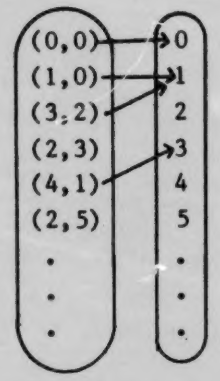


Fig. 2

The assignment is not a mapping. The ordered pair (3,2) → 1, because 3-2 = 1, but the ordered pair (2,3) → ___, because 2-3 is not a whole number. This should help you to understand the importance of expressing the domain in ordered pairs. If the pairs were not ordered and (2,3) was the same as (3,2) then 2-3 would equal 3-2. You already know that this is not true. Therefore (2,3) is not the same as (3,2); the pairs must indeed be ordered. Division is also not a mapping for the same reason; such pairs as (3,4) have no image in W.

If subtraction and division are to define mappings, the set of whole numbers will have to be expanded so that each ordered pair will have an image. Before making any such major changes in the set W, a name should be given to those mappings that already exist. Addition and multiplication are called binary operations on the set of whole numbers. The word binary is used because the operation combines two elements of the domain. Formally, a binary operation is a mapping of ordered pairs of elements of W to W. Expressed in symbols:

$$(a,b) \to c, \text{ for all } a, b, \text{ and } c \text{ in } W$$

A binary operation together with a set, written (W,+) or (W,•), is called an operational system.

Section 2. Finite Operational Systems - Clock Arithmetic

There are many different sets that can be used in an operational system. Some of the ones that follow will be unfamiliar to most of you. These sets are finite. Look at the clock in Fig. 3. There are 12 elements in this set, just like a regular clock.

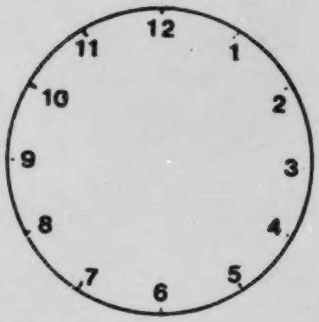


Fig. 3

It is possible to add "clock numbers" together. Sometimes the addition is done in the usual way.

$3 + 4 = 7$ or $(3,4) \rightarrow 7$

$9 + 1 = 10$ or $(9,1) \rightarrow 10$

But what if the sum is greater than 12, for example what is the image of (8,7)? To find 8+7, start at 8 on the clock and count 7 more spaces. You should have ended at 3. So (8,7) → 3 in clock arithmetic. Can you find the image of (9,8)? Use the clock to help you to add 9+8. If you found that (9,8) → 5, you are correct.

Find the image of (12,5) and (9,12). Since 12 + 5 = 5, (12,5) → 5; also 9 + 12 = 9, so (9,12) → 9. If 12 is one of the elements in the ordered pair, (12,a) or (a,12), the image will be a. In whole numbers 0 acts in the same way: (0,5) → 5, since 0 + 5 = 5, and (9,0) → 9, since 9 + 0 = 9. Therefore it is logical to replace the 12 on the clock with 0. Fig. 4 shows the modified clock.

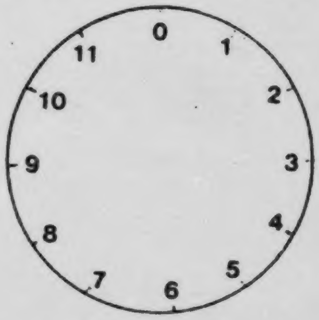


Fig. 4

The twelve elements in the set are now {0,1,2,..., 11}. This is the set Z_{12} (read either as zee-sub-twelve or just zee-twelve). All finite sets are named in this way. The sub-number indicates the number of elements in the set. The symbol Z_n is used to denote this or any finite set having n elements beginning with zero. Finite operational systems are written $(Z_n,+)$, if the binary operation is addition, $(Z_n,\cdot)$, if the operation is multiplication, and so forth.

In the exercises that follow you will make a table to show the sums of all the ordered pairs of elements in Z_{12}. The table form is much easier to use than a list of all 144 ordered pairs in an arrow diagram. (Where did the number 144 come from, you ask. When you complete the table, count all the entries you have made.) In this and in every table, the first element of the ordered pair is along the side and the second element is along the top. When you have completed the table, have your teacher check it before attempting the rest of the exercises.

EXERCISES

1. Complete the $(Z_{12},+)$ table. The 2 in the table is the image of (4,10), not (10,4). Each entry in the table represents the sum a+b.

b

a

+	0	1	2	3	4	5	6	7	8	9	10	11
0												
1												
2												
3												
4											2	
5												
6												
7												
8												
9												
10												
11												

2. Were you able to fill in all the spaces in the table?
Were all the sums unique, that is, was there just one entry in each space?
Is addition a binary operation on Z_{12}?
Is $(Z_{12},+)$ an operational system?

Compute the following sums in Z_{12} arithmetic.

3. 3 + 10 4. 10 + 11 5. 9 + 9
6. 7 + 6 7. 2 + 4 8. 0 + 10

Find the image of each ordered pair in $(Z_{12},+)$.

9. (8,6) → ___ 10. (7,9) → ___ 11. (3,9) → ___
12. (7,7) → ___ 13. (10,11) → ___ 14. (0,0) → ___

To add three numbers in Z_{12} arithmetic, follow this example:

$$\begin{aligned} 4 + (6 + 9) &= 4 + 3, \text{ since } 6 + 9 = 3 \\ &= 7. \end{aligned}$$

Find the following sums in Z_{12}.

15. 3 + (6 + 8) 16. (4 + 9) + 3
17. (2 + 1) + 5 18. (9 + 3) + 6
19. (9 + 9) + 9 20. 5 + (7 + 10)

Section 3. Additive Operational Systems

Another finite system is Z_7, where each number represents one of the days of the week.

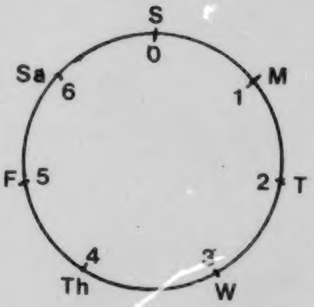


Fig. 1

Addition in Z_7 is very much like addition in Z_{12}. For example, (2,4) → 6, or 2 + 4 = 6 and (5,6) → 4, or 5 + 6 = 4. Find the sums by counting on the clock if you need to, but try to find an easier method if you can.

Some finite systems are very small, Z_2 has only two elements, {0,1}. The addition table for Z_2 appears in Fig. 2.

+	0	1
0	0	1
1	1	0

Fig. 2

Small as it is, $(Z_2,+)$, is an operational system. To be sure you understand this concept, and because the list of ordered pairs is small, an arrow diagram for Z_2 is shown in Fig. 3. The resulting mapping is onto but not one-to-one.

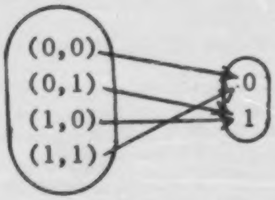


Fig. 3

The exercises that follow will introduce you to some other finite systems. Begin by filling out all the addition tables. Save them. You will need to refer to them later.

EXERCISES

Obtain a copy of these tables from your teacher.

Z_3

+	0	1	2
0			
1			
2			

Z_4

+	0	1	2	3
0				
1				
2				
3				

Z_5

+	0	1	2	3	4
0					
1					
2					
3					
4					

Z_6

+	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

Z_7

+	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Z_8

+	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

Compute the following in Z_7.

1. 2 + 1
2. 6 + 2
3. 4 + 5
4. 5 + 4
5. 0 + 6
6. 5 + 5

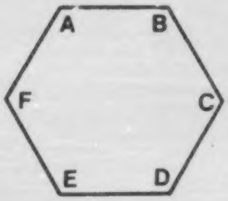
Find each image in $(Z_5,+)$.

7. (2,3) → ___
8. (4,2) → ___
9. (1,3) → ___

Compute in Z_8.

10. 6 + 5
11. 7 + 1
12. 2 + 5
13. (5+7) + 4
14. 6 + (4+2)
15. 4 + (3+6)
16. (5+3) + 2

17. Suppose a clock had letters instead of numbers. Construct the addition table for the clock, using the letters shown in Fig. 4. Notice that it has the same number of elements as Z_6. Use your Z_6 table to help you find the sums.



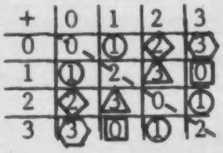
+	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

In this arithmetic find

18. A + F
19. C + C
20. D + F
21. D + E

Section 4. Properties of Operational Systems

Some of the exercises you have just completed illustrate certain important properties of addition. One of the simplest, the commutative property, states that a + b = b + a, or (a,b) = (b,a), for all elements a and b in the operational system. Study the $(Z_4,+)$ table in Fig. 1.



+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Fig. 1

All of the ordered pairs that commute are indicated by symbols. For example, since (1,3) = (3,1), the sum of each ordered pair, 0, is enclosed in a box. Lay your pencil along the dashed line. This diagonal splits the table into two halves that have identical entries in corresponding positions. Try this simple test for the commutative property on the other tables you have constructed. In all finite systems, as well as in the set of whole numbers, addition is a commutative operation.

Another property that can be verified directly from the addition table involves zero. In Fig. 2, the only sums entered are those where 0 is one of the elements of the ordered pair.

+	0	1	2	3
0	0	1	2	3
1	1			
2	2			
3	3			

Fig. 2

The row and column headed by 0 are the same as the outer row and column. This special property of 0 can be found in any addition table. For this reason, zero is called the identity element for addition. In every additive operational system, $a + 0 = a$ and $0 + a = a$; or $(a,0) \to a$ and $(0,a) \to a$, for all elements a in the system.

A third property evident from the table is the inverse property. Two elements whose sum is the identity element, 0, are called additive inverses of each other; that is, a and b are inverses if $a + b = 0$ or $(a,b) \to 0$. In $(Z_4,+)$, $(1,3) \to 0$ and $(3,1) \to 0$, so 1 and 3 are inverses of each other. Like the identity element, addition of inverses is always commutative. The other two elements in Z_4 are their own inverses, $(0,0) \to 0$, and $(2,2) \to 0$. Because each element has an inverse, Z_4 has the inverse property for addition. To see how to find inverses directly from the table, study the partial $(Z_4,+)$ table in Fig. 3.

+	0	1	2	3
0	0			
1				0
2			0	
3		0		

Fig. 3

Since the identity element occurs in the first row, there must be an element to complete the ordered pair $(0,\_) \to 0$. The zero in the second row shows that an element exists to complete the ordered pair $(1,\_) \to 0$. The same is true in the other two rows. Check each column. Is there a 0 in each column? Yes. Then each pair $(\_,a) \to 0$ can be completed. If the identity element appears in each row and each column, then every element in the operational system has an inverse, and the system itself has the inverse property.

To indicate the additive inverse of an element, place a long bar in front of it. The additive inverse of 3 is written -3. In other words, $(3,-3) \to 0$, or $(a,-a) \to 0$, where a represents any element in the system. You just learned that in Z_4, $(3,1) \to 0$. Since $(3,-3) \to 0$, $-3 = 1$. A complete list of the additive inverses in Z_4 follows.

$(0,-0) \to 0$ and $(0,0) \to 0$ so $-0 = 0$
$(1,-1) \to 0$ and $(1,3) \to 0$ so $-1 = 3$
$(2,-2) \to 0$ and $(2,2) \to 0$ so $-2 = 2$
$(3,-3) \to 0$ and $(3,1) \to 0$ so $-3 = 1$

The additive inverses of each element in $(Z_5,+)$ are shown below, using addition symbols instead of mapping symbols. You should be able to use both types of notation. Remember that the sum of a number and its additive inverse is equal to zero. The ()'s are placed around the $-a$ so there will be no confusion with the symbol for addition.

$0 + (-0) = 0$ and $0 + 0 = 0$ so $-0 = 0$
$1 + (-1) = 0$ and $1 + 4 = 0$ so $-1 = 4$
$2 + (-2) = 0$ and $2 + 3 = 0$ so $-2 = 3$
$3 + (-3) = 0$ and $3 + 2 = 0$ so $-3 = 2$
$4 + (-4) = 0$ and $4 + 1 = 0$ so $-4 = 1$

It is possible to simplify expressions involving the additive inverse symbol. Here are several examples in $(Z_5,+)$.

Ex. 1 $(-3) + 4$
$= 2 + 4$
$= 1$

Ex. 2 $(-2) + (-3)$
$= 3 + 2$
$= 0$

Ex. 3 $(4, -3) \to$ ____
$(4, 2) \to 1$

Ex. 4 $-(2 + 1)$
$= -3$
$= 2$

Ex. 5 $-3 + (2 + (-4))$
$= 2 + (2 + 1)$
$= 2 + 3$
$= 0$

The exercises that follow will give you practice finding additive inverses and computing with them.

One property that cannot be established from the addition table by inspection is the associative property. An operational system is associative if $(a+b)+c = a+(b+c)$, for all a, b and c in the system. The parentheses ( ) are necessary because addition is a binary operation and therefore combines only two elements at a time. The parentheses indicate which pair to combine first.

Consider $(Z_5,+)$. One example of the associative property would be

$$\begin{gathered} 3 + (2 + 1) = (3 + 2) + 1 \\ 3 + 3 = 0 + 1 \\ 1 = 1 \end{gathered}$$

There is no way to be certain that a system is associative except to list all the examples possible and to check each one to be sure that the expression on the left of the equal sign is equivalent to the expression on the right. The list is very long for most systems so you should either choose a system with very few elements or work together with someone else. Usually, you will find that it is sufficient to test only a few examples before declaring the system to be associative. There are some notable exceptions which you will meet later.

All $(Z_n,+)$ have four common properties: the commutative property, an identity element, the inverse property, and the associative property. Other operational systems have some or all of these same properties. The following important definition will help distinguish among the various types of operational systems.

A GROUP IS AN OPERATIONAL SYSTEM THAT HAS:
1) THE ASSOCIATIVE PROPERTY
2) AN IDENTITY ELEMENT
3) THE INVERSE PROPERTY

If the group is also commutative, it is called a commutative group. All $(Z_n,+)$ are commutative groups. However, not all groups are commutative, as you will soon see.

One important operational system is not a group. Addition on the set of whole numbers is associative. A few examples of your own should convince you of this. There is an identity element, namely 0. But what is the additive inverse of 4? In other words, what whole number correctly completes the ordered pair (4,_) → 0? There is no whole number for which 4 + _ = 0; therefore (W,+) does not have the inverse property and is not a group.

EXERCISES

Find the additive inverse for each element in $(Z_2,+)$ through $(Z_9,+)$. Use the format shown below.

$(Z_2,+)$
-0 = ___
-1 = ___

$(Z_3,+)$
-0 = ___
-1 = ___
-2 = ___

$(Z_4,+)$
-0 = ___
-1 = ___
-2 = ___
-3 = ___

etc.

Compute in $(Z_7,+)$

1. (-3)+4   2. (-4)+5   3. (-3)+(-4)

4. 1 + (-6)   5. (2+3)+(-5)   6. 3+((-2)+2)

Compute in $(Z_9,+)$

7. (-6)+(-7)   8. 5+(-8)   9. (4+8)+(-6)

10. (4+(-8))+3   11. -(6+5)   12. -(4+(-5))

In $(Z_7,+)$ compute the following

13. (-3)+(-4)   14. -(3+4)   15. (-5)+(-2)

16. -(5+2)   17. (-4)+(-1)   18. -(4+1)

19. -(-3)+(-5)   20. -((-3)+5)

21. Complete this statement: For all $(Z_n,+)$
-(a+b) = ___________

Compute the following in $(Z_8,+)$

22. -(-5)   23. -(-7)   24. -(-2)

25. Complete this statement: For all $(Z_n,+)$
-(-a) = ________

Simplify these expressions in $(Z_9,+)$
26. -(-(-3))   27. -(-(-(-6)))   28. -(-(-(-(-7))))

29. Do you see a pattern? Write a rule showing the relationship between the number of symbols (-) and the final value of the expression.

Which of the following correctly illustrates the associative property? Examples are in $(Z_9,+)$.

30. $(6+2)+1 = 6+(2+1)$
31. $7+(2+5) = (2+5)+7$
32. $(5+4)+3 = (4+5)+3$
33. $6+(4+2) = (6+4)+2$
34. $(5+2)+7 = (5+7)+2$
35. $4+(6+3) = (4+3)+6$

35. All $(Z_n,+)$ are commutative groups and therefore have 4 properties: associative property, identity element, inverse property and commutative property. Give an example of each of these properties using elements from $(Z_8,+)$.

Each statement below is an example of one of the four properties listed in problem 35. Identify which property is illustrated by each statement in $(Z_9,+)$.

37. $3+5 = 5+3$
38. $4+(-4) = 0$
39. $6+(3+2) = (6+3)+2$
40. $3 + 0 = 3$
41. $(6+(-6))+2 = 0+2$
42. $(5+3)+(-3) = 5+(3+(-3))$
43. $(6+2)+(-5) = (2+6)+(-5)$
44. $(3+(-4))+2 = 2+(3+(-4))$
45. $(7+(-7))+2 = 2+(7+(-7))$
46. $(7+(-7))+2 = 0+2$
47. $0+(-3)=(-3)$
48. $(1+0)+1 = 1+1$

Section 5. Multiplicative Systems.

Another binary operation on the set of whole numbers is multiplication. $(W,\cdot)$ is an operational system since every ordered pair of elements in the set of whole numbers, W, has an image in the same set. Does $(W,\cdot)$ form a group? Checking a few examples is sufficient to indicate that $(W,\cdot)$ is associative. The identity element for multiplication is not 0 as in addition, because $(a,0) \to 0$ or $a\cdot 0 = 0$. Recall that if a member of the set is to be the identity element, then (a,identity element) → a for all a in the set. The obvious choice for the multiplicative identity element is 1, since

$$(1,a) \to 1 \text{ and } (a,1) \to 1 \text{ or}$$
$$1\cdot a = a \text{ and } a\cdot 1 = a, \text{ for all } a \text{ in } W.$$

It remains only to establish the inverse property before declaring $(W,\cdot)$ a group. What is the multiplicative inverse of 4? Unfortunately there is no whole number to correctly complete the sentence $4 \cdot \_\_\_ = 1$. Therefore $(W,\cdot)$ is not a group because it does not have the inverse property.

It may be possible that some or all of the finite operational systems $(Z_n, \cdot)$ form multiplicative groups. Since several properties are evident in the tables themselves, the first thing that needs to be done is to make out multiplication tables for the various operational systems. Fill out the tables at the beginning of the exercises.

The patterns are quite different from addition, so be very careful. Was it possible to find a unique product for every pair of elements? Yes, then the necessary mapping exists and each table defines an operational system. The three properties of a group must now be tested. Look at the $(Z_5, \cdot)$ table:

•	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Fig. 1

The first property to test is the associative property. Here are two examples:

Ex. 1.

$$(3 \cdot 4) \cdot 2 = 3 \cdot (4 \cdot 2)$$
$$2 \cdot 2 = 3 \cdot 3$$
$$4 = 4$$

Ex. 2.

$$(2 \cdot 4) \cdot 4 = 2 \cdot (4 \cdot 4)$$
$$3 \cdot 4 = 2 \cdot 1$$
$$2 = 2$$

Try a few other examples yourself to be certain that the system is associative.

To determine the identity element, find the row and column that are just like the outer row and column. The row and column headed by 1 satisfy this condition. Therefore, 1 is the identity element. Writing out the various examples as a check gives

$$1 \cdot 0 = 0 \cdot 1 = 0$$
$$1 \cdot 1 = 1 \cdot 1 = 1$$
$$1 \cdot 2 = 2 \cdot 1 = 2$$
$$1 \cdot 3 = 3 \cdot 1 = 3$$
$$1 \cdot 4 = 4 \cdot 1 = 4$$

Further, does each element have an inverse; in other words is there a number to complete the ordered pairs $(a,\_\_) \to 1$ and $(\_\_,a) \to 1$ for all a in Z_5? Most pairs can be completed easily.

$$(1,1) \to 1;\ 1 \cdot 1 = 1$$
$$(2,3) \to 1;\ 2 \cdot 3 = 1$$
$$(3,2) \to 1;\ 3 \cdot 2 = 1$$
$$(4,4) \to 1;\ 4 \cdot 4 = 1$$

Therefore 2 and 3 are inverses of each other, while 1 and 4 are their own inverses.

But consider 0; $(0,\_\_) \to 1$ means $0 \cdot \_\_\_ = 1$. There is no number that can complete the ordered pair. The product of any number and 0 is always 0, never 1. Zero, therefore, has no multiplicative inverse.

In addition the inverse property could be checked by locating the identity element in each row and column of the table. The following partial table for $(Z_5, \cdot)$ shows only the placement of the identity element 1.

•	0	1	2	3	4
0					
1		1			
2				1	
3			1		
4					1

Fig. 2

Notice that there is no 1 in the row and column headed by 0. This method, too, shows that zero has no inverse.

This same difficulty with 0 will be found in every multiplicative system. Zero never has a multiplicative inverse. Frequently all the other elements have inverses, and, like $(Z_5, \cdot)$, if 0 were removed the system would be a group. So, for operational systems involving multiplication, only the non-zero elements will be considered. To indicate that 0 is suppressed, write $Z_n \setminus \{0\}$, which is read as Z sub n without 0. The operational system now becomes $(Z_5 \setminus \{0\}, \cdot)$ and is a group. A check of the table shows that the two sides of the diagonal are identical so that the group is commutative.

Now consider the table for $(Z_6\setminus\{0\},\cdot)$:

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Fig. 3

The check for the associative property is left to you. As in all multiplicative systems, the identity element is one; the row and column headed by one are the same as the outer row and column. To find the multiplicative inverses, locate the 1's in the table. There are only two; $1\cdot1 = 1$ and $5\cdot5 = 1$. Therefore 1 and 5 are their own inverses, but none of the other elements have inverses. Check the row headed by 2. The only products possible are 0, 2 or 4. There is no way to complete the expression $2\cdot\_\_\_ = 1$. The same is true for 3 and 4. So even if 0 is suppressed $(Z_6\setminus\{0\},\cdot)$ is not a group.

In addition, all Z_n are groups. This is obviously not the case for multiplication. $(Z_5\setminus\{0\},\cdot)$ is a group; $(Z_6\setminus\{0\},\cdot)$ is not. In the exercises that follow you will be asked to investigate other multiplicative systems, separating the groups from the non-groups. A careful analysis will enable you to state a rule that determines which systems are groups and which are not.

Just as there is a special symbol for the additive inverse, there is one for the multiplicative inverse. The multiplicative inverse of a is written a^{-1}. Using this new notation, the inverse property for multiplication can be restated as follows:

$$(a,a^{-1}) \rightarrow 1 \text{ and } (a^{-1},a) \rightarrow 1$$

$$\text{or } a\cdot a^{-1} = 1 \text{ and } a^{-1}\cdot a = 1, \text{ for all a in the set.}$$

Applying this definition to $(Z_5\setminus\{0\},\cdot)$:

$(1,1^{-1}) \rightarrow 1$ and $(1,1) \rightarrow 1$, so $1^{-1} = 1$
$(2,2^{-1}) \rightarrow 1$ and $(2,3) \rightarrow 1$, so $2^{-1} = 3$
$(3,3^{-1}) \rightarrow 1$ and $(3,2) \rightarrow 1$, so $3^{-1} = 2$
$(4,4^{-1}) \rightarrow 1$ and $(4,4) \rightarrow 1$, so $4^{-1} = 4$

It is also possible to simplify expressions involving multiplicative inverses. Some examples in Z_5 follow:

Ex. 1.

$$4 \cdot 3^{-1}$$
$$= 4 \cdot 2$$
$$= 3$$

Ex. 2.

$$2^{-1} \cdot 4$$
$$= 3 \cdot 4$$
$$= 2$$

Ex. 3.

$$(4, 2^{-1}) \to \underline{\quad}$$
$$(4, 3) \to 2$$

EXERCISES

Complete these tables.

Z_3

$\cdot$	0	1	2
0			
1			
2			

Z_4

$\cdot$	0	1	2	3
0				
1				
2				
3				

Z_5

$\cdot$	0	1	2	3	4
0					
1					
2					
3					
4					

Z_6

$\cdot$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

Z_7

$\cdot$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Z_8

$\cdot$	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

1. Find the multiplicative inverse of each element in $(Z_2\setminus\{0\},\cdot)$ through $(Z_9\setminus\{0\},\cdot)$. If no inverse exists write none. Use the format shown below.

$(Z_2\setminus\{0\},\cdot)$

1^{-1} = ___

$(Z_3\setminus\{0\},\cdot)$

1^{-1} = ___
2^{-1} = ___

$(Z_4\setminus\{0\},\cdot)$

1^{-1} = ___
2^{-1} = ___
3^{-1} = ___

$(Z_5\setminus\{0\},\cdot)$

1^{-1} = ___
2^{-1} = ___
3^{-1} = ___
4^{-1} = ___

etc.

Simplify each expression in Z_7:

2. $6^{-1}\cdot 3$
3. $2^{-1}\cdot 4^{-1}$
4. $5^{-1}\cdot 2$
5. $3\cdot 4^{-1}$
6. $3^{-1}\cdot(2^{-1}\cdot 4)$
7. $4^{-1}\cdot(3^{-1}\cdot 2)$
8. $(3^{-1}\cdot 6^{-1})\cdot 2$
9. $(3^{-1}\cdot 2)\cdot 0$
10. $(5^{-1}\cdot 6)\cdot 3^{-1}$
11. $(6\cdot 6^{-1})\cdot 4$
12. $2\cdot(4^{-1}\cdot 3^{-1})$

Find the image of each ordered pair in $(Z_9\setminus\{0\},\cdot)$:

13. $(2^{-1},4)\rightarrow$ ___

14. $(5^{-1},7^{-1})\rightarrow$ ___

15. $(6^{-1},3)\rightarrow$ ___

16. $(7^{-1},2)\rightarrow$ ___

17. $(4^{-1},4)\rightarrow$ ___

18. $(5^{-1},0)\rightarrow$ ___

19. The ordered pair in 15 has no image. Why?

Recall that in addition $-(a+b) = (-a) + (-b)$. To determine if a similar rule can be written for multiplication, compute the following in $(Z_7\setminus\{0\},\cdot)$.

20. $(3\cdot4)^{-1}$

21. $3^{-1}\cdot4^{-1}$

22. $(5\cdot2)^{-1}$

23. $5^{-1}\cdot2^{-1}$

24. $(4\cdot1)^{-1}$

25. $4^{-1}\cdot1^{-1}$

26. $(5\cdot3^{-1})^{-1}$

28. $5^{-1}\cdot(3^{-1})^{-1}$

28. Complete this statement: For multiplicative groups $(a\cdot b)^{-1} =$ ____

In addition $-(-a) = a$. Can this rule be extended to multiplication? Simplify these expressions in $(Z_7\setminus\{0\},\cdot)$

29. $(3^{-1})^{-1}$

30. $(4^{-1})^{-1}$

31. $(5^{-1})^{-1}$

32. Complete this statement: For multiplicative groups $(a^{-1})^{-1} =$ ____.

Simplify these expressions in $(Z_7\setminus\{0\},\cdot)$

33. $(6^{-1})^{-1}$

34. $((6^{-1})^{-1})^{-1}$

35. $(((5^{-1})^{-1})^{-1})^{-1}$

36. $((((2^{-1})^{-1})^{-1})^{-1})^{-1}$

37. State a rule for the pattern shown in problems 33-36.

38. Which $(Z_n\setminus\{0\},\cdot)$ are commutative groups? List the groups on one line and the non-groups on another line.

39. Compare the two lists from problem 38. The set of natural numbers is separated in many ways, for example into odd numbers and even numbers. Do all the groups have an odd number of elements and the non-groups an even number of elements? If not, what two terms do you know that correctly describe the number of elements in all the sets that are groups and all the sets that are not groups?

40. Complete this statement: $(Z_n \setminus \{0\}, \cdot)$ is a commutative group if the number of elements in the set n is a _________ number.

41. Using this rule, is $(Z_{12} \setminus \{0\}, \cdot)$ a commutative group?

Section 6. Subtraction and Division

There are two other operations that you are familiar with - subtraction and division. Is either a binary operation on the set of whole numbers? Recall the definition of a binary operation: a binary operation is a mapping of ordered pairs of elements of a set to elements in the same set. In symbols

$$(a,b) \rightarrow c, \quad \text{for all a, b and c in the set.}$$

In subtraction many pairs of whole numbers have unique solutions, $(7,3) \rightarrow 4$; $(19,3) \rightarrow 16$, etc. However, many pairs do not have images in the set of whole numbers, W, for example, (3,5) and (7,12). The ordered pairs (a,b) have images only if a is greater than or equal to b. Because of this limitation, subtraction is called a restricted operation on the set of whole numbers. Division of whole numbers is also restricted to those pairs of elements (a,b) where b divides a evenly with no remainder.

Is subtraction a binary operation on the finite number systems? To answer this question, tables can be constructed that are similar to those for addition and multiplication. Remember that the number in the left column is the first element of the pair. The table below for $(Z_5, -)$ shows the differences for pairs where a is greater than or equal to b.

-	0	1	2	3	4
0	0				
1	1	0			
2	2	1	0		
3	3	2	1	0	
4	4	3	2	1	0

For pairs where a is less than b think of a clock. For example (2,3). To subtract 3 from 2, begin at 2 and count backwards 3. You should have ended at 4; so $2 - 3 = 4$. The table can be completed in this way. The same result can also be reached without the clock. In arithmetic, you learned to check subtraction problems by addition. For example,

$$\text{if} \quad \begin{array}{r} 963 \\ -481 \\ \hline 482 \end{array} \qquad \text{then} \quad \begin{array}{r} 482 \\ +481 \\ \hline 963 \end{array}$$

Using this relationship, subtraction can be defined as follows

$$\text{if } a-b = c \text{ then } c+b = a$$

In Z_5 if $4-1 = 3$, then $3+1 = 4$. Also if $4-2 \neq 3$, then $3 + 2 \neq 4$. The two parts of the statement are either both true or both false. Applying this to the problem 2-3:

$$\text{if } 2-3 = 4, \text{ then } 4 + 3 = 2.$$

Since $4+3 = 2$ is true, the subtraction must also be correct, that is, the problem "checks". Here is another example. Subtract one from four.

1. Let $1-4 = c$
2. Write as an addition problem $c+4 = 1$
3. Solve the addition problem $c = 2$
4. Then $1-4 = 2$

Ex. 1 Compute 3-4:

$$\begin{aligned} 3-4 &= c \\ c+4 &= 3 \\ c &= 4 \\ 3-4 &= 4 \end{aligned}$$

Ex. 2 Compute 2-4:

$$\begin{aligned} 2-4 &= c \\ c+4 &= 2 \\ c &= 3 \\ 2-4 &= 3 \end{aligned}$$

The completed table for $(Z_5,-)$ appears below.

-	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Does the table illustrate an operational system? Yes, each ordered pair has an image in the set Z_5. To determine if the system is a group, first check the associative property:

$$3-(2-4) = (3-2)-4$$
$$3 - 3 = 1 - 4$$
$$0 \neq 2$$

Since the left half and the right half of the statement are not equal, the system is not associative, and therefore not a group.

Is there an identity element? The column under 0 is the same as the outer column, but the row across from 0 is not the same as the top row. This means that $a-0 = a$ but $0-a \neq a$. For 0 to be the identity element, both of these statements must be true. So $(Z_5,-)$ has no identity element, and therefore no inverse property. The system is not even commutative. Check this.

All finite systems are similar to $(Z_5,-)$. They are operational systems but do not have any of the group properties.

Division in finite systems is even more unusual. To complete tables for division, recall that in whole numbers, division problems could be checked by multiplication. For example, if $18 \div 2 = 9$, then $9 \cdot 2 = 18$. This relationship can be used to define division in terms of multiplication in the same way that subtraction is defined in terms of addition.

If $a \div b = c$, then $c \cdot b = a$

Apply this to the problem $2 \div 3$ in $(Z_5,\div)$ in this way.

1. Let $2 \div 3 = c$
2. Then $c \cdot 3 = 2$
3. So $c = 4$
4. Then $2 \div 3 = 4$

Study these examples in $(Z_5, \div)$

Ex. 3
$4 \div 3 = c$
$c \cdot 3 = 4$
$c = 3$
$4 \div 3 = 3$

Ex. 4
$2 \div 4 = c$
$c \cdot 4 = 2$
$c = 3$
$2 \div 4 = 3$

Ex. 5
$0 \div 4 = c$
$c \cdot 4 = 0$
$c = 0$
$0 \div 4 = 0$

Ex. 6
$3 \div 0 = c$
$c \cdot 0 = 3$
no solution

Look at example 6 again. Because there is no solution to the multiplication problem $c \cdot 0 = 3$, there is no solution to the division problem $3 \div 0$. Division by zero is undefined. Zero may not be used as a divisor. Zero may be the dividend however, as shown in example 5, where $0 \div 4 = 0$.

The table for $(Z_5, \div)$ is completed by using the method outlined above.

÷	0	1	2	3	4
0	-	0	0	0	0
1	-	1	3	2	4
2	-	2	1	4	3
3	-	3	4	1	2
4	-	4	2	3	1

The table does not illustrate an operational system unless 0 is suppressed. Once the row and column headed by zero are removed the remaining elements do form an operational system. As in subtraction the system is not associative as shown by the following example.

$$3 \div (4 \div 2) = (3 \div 4) \div 2$$
$$3 \div 2 = 2 \div 2$$
$$4 \neq 1$$

There is also no identity element. The column under one is the same as the outer column but the row across from one is not the same as the outer row. For example, $3 \div 1 = 3$ but $1 \div 3 \neq 3$, rather $1 \div 3 = 2$. The system therefore has no inverse property. It also lacks commutativity. Notice that the situation exactly parallels that found in subtraction. You might conclude therefore, that all $(Z_n \setminus \{0\}, \div)$ are operational systems but have none of the group properties. Before coming to this decision recall these facts. All $(Z_n, +)$ are commutative groups and all $(Z_n, -)$ are operational systems. All $(Z_n \setminus \{0\}, \cdot)$ are not commutative groups. Only those where n is prime are groups. For the system above, $(Z_5 \setminus \{0\}, \div)$, the related multiplication system is a commutative group because 5 is a prime number. What happens, then, in $(Z_6 \setminus \{0\}, \div)$ when the related multiplication system is not a group? In making a table for $(Z_6 \setminus \{0\}, \div)$ some unusual situations arise.

Ex. 7 $3 \div 2 = c$
$c \cdot 2 = 3$

There is no solution for this statement in Z_6 multiplication. In Z_5 the only division problems that had no solution were those where 0 was the divisor. Several other problems in Z_6 have no solution.

Ex. 8 $4 \div 3 = c$
$c \cdot 3 = 4$

Ex. 9 $2 \div 3 = c$
$c \cdot 3 = 2$

Ex. 10 $5 \div 4 = c$
$c \cdot 4 = 5$

Be certain that you can explain why each of these problems has no solution.

Now consider this problem

$$2 \div 4 = c$$
$$c \cdot 4 = 2$$

This multiplication problem has two solutions, 2 and 5, since

$$2 \cdot 4 = 2 \text{ and } 5 \cdot 4 = 2$$

Other problems also have more than one solution.

Ex. 11 $0 \div 3 = c$
$c \cdot 3 = 0$
0,2,4

Ex. 12 $4 \div 2 = c$
$c \cdot 2 = 4$
2,5

Combining all these problems gives this table for $(Z_6,\div)$

÷	0	1	2	3	4	5
0	-	0	0,3	0,2,4	0,3	0
1	-	1	-	-	-	5
2	-	2	1,4	-	2,5	4
3	-	3	-	1,3,5	-	3
4	-	4	2,5	-	1,4	2
5	-	5	-	-	-	1

Does this table illustrate an operational system? Certainly not. Most pairs do not have unique images in the set. Therefore, for division, the only finite sets that are operational systems are those where the number of elements is prime and, of course, only when zero is suppressed.

EXERCISES

1. Make a table for $(Z_6,-)$ and $(Z_8\backslash\{0\},\div)$

Compute in $(Z_6,-)$
2. 1-(5-2)
3. (1-5)-2
4. (1-4)-2
5. 1-(4-2)
6. 5-(2-4)
7. (2-3)-5

Compute in $(Z_8\backslash\{0\},\div)$
8. 7÷3
9. 6÷4
10. 0÷2
11. (3÷5)÷3
12. 3÷(5÷7)
13. (2÷5)÷7

REVIEW FINITE NUMBER SYSTEMS

1. Terms to define:
   binary operation
   operational system
   group
   These properties for both addition and multiplication:
      A) Associative Property
      B) Commutative Property
      C) Identity Element
      D) Inverse Property

2. Which of the following are operational systems? $(Z_n,+)$, $(Z_n\backslash\{0\},\cdot)$, $(Z_n,-)$, $(Z_n\backslash\{0\},\div)$

3. Make a table for any $(Z_n,+)$, $(Z_n\backslash\{0\},\cdot)$, $(Z_n,-)$, $(Z_n\backslash\{0\},\div)$ that is an operational system.

4. Which systems in problem 3 are commutative groups? Identify the identity element and list all inverses. For each group give examples of the associative property.

5. Explain how to locate the identity element and inverses from the table of an operational system. Explain how to check for the commutative property directly from the table. Explain why the associative property cannot be checked in this way.

6. Perform the following computations in Z_8.

a) $3^{-1}\cdot 5$
b) $(-2) + (-3)$
c) $5^{-1}\cdot(5^{-1}\cdot 1^{-1})$
d) $7^{-1}\cdot 7^{-1}$
e) $(-2) + -(-4)$
f) $-7 + ((-3) + (-4))$
g) $-(2+(-4))$
h) $3 \div 5$
i) $((-2) + (-5)) + (-3)$
j) $1 \div (3 \div 7)$
k) $1 - (3 - 7)$
l) $(1-5) - 2$
m) $2 \div 6$
n) $5 \div 4$
o) $(7^{-1}\cdot 3^{-1})\cdot 5^{-1}$

Answers

For question 1-5, see text.
6. a)7, b) 3, c) 1, d) 1, e) 2, f) 2, g) 2, h) 7, i) 6, j) 5, k) 5, l) 2
m) 3,7, n) no solution, o) 1

PRE-ALGEBRA GROUPS

Chapter 3

Other Groups

Chapter 3 Other Groups

Section 1. Other Operational Systems

In this section you will have a chance to explore other operational systems. The set, or the binary operation, or both may be unfamiliar to you. Some systems will be groups, others will not. Until now you have defined a group using two sets of symbols, one for addition and one for multiplication. It is useful to have a single more general definition of a group to fit all sets and binary operations. The set is called S and the binary operation *. Each property is defined using elements a, b, and c in S. The identity element is called e and a^I is the inverse of a. Here then is the complete definition.

A group is an operational system (S,*) which has

1) the associative property
   $(a*b)*c = a*(b*c)$, for all a, b and c in S.
2) an identity element e
   $a*e = e*a = a$, for all a in S.
3) the inverse property
   $a*a^I = a^I*a = e$, for all a in S.

The first operational system to be considered is (E,d). The set E={2, 4, 6, 8}. The operation "d" is digital multiplication. In ordinary arithmetic $4 \cdot 8 = 32$; however, in digital multiplication only the last digit is used, thus 4 d 8 = 2. Also 4 d 6 = 4 since $4 \cdot 6 = 24$, drop the 2, keeping just the 4. Figure 1 shows the complete table for the operational system (E,d).

d	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

Fig. 1

Do you think (E,d) is associative? Here are two examples. Be sure to justify each step yourself.

Ex. 1. (4 d 8) d 2 = 4 d (8 d 2)
2 d 2 = 4 d 6
4 = 4

Ex. 2. $(6 \text{ d } 2) \text{ d } 8 = 6 \text{ d } (2 \text{ d } 8)$

$$2 \text{ d } 8 = 6 \text{ d } 6$$

$$6 = 6$$

You should write out several more examples before continuing. Is there an identity element; can you find a row and column just like the outside row and column? The number at the head of that row and column is the identity element. For this operational system it is 6. Notice that unlike addition and multiplication, the identity element is not the first element in the table.

Is there an inverse for each element? Remember that to answer yes, you must locate the identity element 6 in each row and in each column. The elements 2 and 8 are inverses of each other since (2,8) → 6 and (8,2) → 6 while 4 and 6 are their own inverses since (4,4) → 6 and (6,6) → 6. Therefore (E,d) is a group. In fact it is a commutative group. Check this.

Sometimes the elements of a system are letters instead of numbers, and the operation is defined on the set only by the given table. Consider A = {a, b, c} with the operation # defined on A by the table in Fig. 2.

#	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

Fig. 2

To determine if the system is a group, check each property. Is it associative?

Ex. 1.

$$(a \# c) \# a = a \# (c \# a)$$

$$b \# a = a \# b$$

$$a = a$$

Ex. 2.

$$(b \# c) \# a = b \# (c \# a)$$

$$c \# a = b \# b$$

$$b = b$$

It appears to be associative.

Is there an identity element? Yes, b is the identity element. Each element has an inverse since b occurs in each row and column. The system is also commutative and thus forms a commutative group.

In the third and last example, R = {a,b,c,d}. The operation @ is defined on R by the table below:

@	a	b	c	d
a	a	b	c	d
b	c	a	d	b
c	b	d	a	c
d	d	c	b	a

The system appears to be associative.

Ex. 1.

$$
\begin{gathered}
(a \mathbin{@} c) \mathbin{@} b = a \mathbin{@} (c \mathbin{@} b) \\
c \mathbin{@} b = a \mathbin{@} d \\
d = d
\end{gathered}
$$

Ex. 2.

$$
\begin{gathered}
(d \mathbin{@} c) \mathbin{@} a = d \mathbin{@} (c \mathbin{@} a) \\
b \mathbin{@} a = d \mathbin{@} b \\
c = c
\end{gathered}
$$

What is the identity element? The row opposite a is like the row at the top, but the column under a is not the same as the outer column. For example, $a \mathbin{@} c = c$; but $c \mathbin{@} a \neq c$, instead $c \mathbin{@} a = b$. Therefore there is no identity element, and the system is not a group. It is not even possible to check the inverse property since there is no identity element. Perhaps the operational system is not even associative.

$$
\begin{gathered}
(c \mathbin{@} b) \mathbin{@} d = c \mathbin{@} (b \mathbin{@} d) \\
d \mathbin{@} d = c \mathbin{@} b \\
a \neq d
\end{gathered}
$$

Obviously $a \neq d$ so the system is not associative. Notice however, that the two earlier examples made it appear that (R,@) was associative when really it was not. This is why it is necessary to check all the possible combinations of elements before you can be absolutely certain whether or not a system is associative.

The exercises that follow will introduce you to some other operational systems, some of which will prove to be groups.

EXERCISES

1. Given the set P = {1, 2, 3, 4, 5, 6} and binary operation ∇ defined on P by the table below, determine whether or not the system is a group.

∇	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

a) Is (P,∇) associative? Give two examples.

b) Is there an identity element? Identify it.

c) Does each element have an inverse? List the inverse of each element if one exists.

d) Is (P,∇) a group?

e) Is (P,∇) commutative?

2. Use the operation d, digital multiplication, as defined earlier and the set G = {1, 3, 7, 9}. Complete the table for (G, d).

d	1	3	7	9
1				
3				
7				
9				

a) Is the system associative? Give two examples.

b) Is there an identity element? Which element is it?

c) List the inverse of each element if one exists.

d) Is (G,d) a group? Is it a commutative group?

e) Why do you think 5 was not included in G? Make a table using {1, 3, 5, 7, 9} in order to answer this question.

3. Given the set F = {1, 2, 3, 4, 6, 12}, construct a table using the binary operation f where f is the greatest common factor of each pair. For example, 6 f 3 = 3 since 3 is the largest number that divides both 6 and 3 evenly; and 3 f 4 = 1 since only 1 divides both 3 and 4.

f	1	2	3	4	6	12
1						
2						
3						
4						
6						
12						

Determine whether or not the system is a commutative group:
a)Is it associative? Give two examples.
b)Is there an identity element? Identify it.
c)List the inverse of each element if one exists.
d)Is (F,f) a group?
e)Is the system commutative?
f)Is (F,f) a commutative group?

4. Now go back and, starting in Chapter 2, list all the operational systems you can that are groups. Which of these are commutative groups? List all the operational systems that are not groups. Use the headings shown below.

GROUPS	COMMUTATIVE?	NON-GROUPS

Section 2. Cancellation Law

There is an important conclusion to be drawn from the work you have just completed. Consider the following problem in $(Z_5,+)$. Some number a is added to 4 giving the sum 4+a; a number b is also added to 4 giving 4+b. The two sums are compared and found to be equal, that is $4+a = 4+b$. Can any conclusion be drawn about the value of a and b? If you concluded that $a = b$, you are correct. In $(Z_5,+)$ if $4+a = 4+b$, then $a = b$. Now do the problem in $(Z_6\setminus\{0\},\cdot)$. Multiplying 4 times a gives $4\cdot a$, and 4 times b gives $4\cdot b$. If the products are the same, that is $4\cdot a = 4\cdot b$, is it correct to conclude that $a = b$? It is not.

This important difference between $(Z_5,+)$ and $(Z_6\setminus\{0\},\cdot)$ can be seen in the tables.

$(Z_5,+)$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$(Z_6\setminus\{0\},\cdot)$

•	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Look carefully at the row beginning with 4 in both tables. Do you notice any differences? The row in the Z_5 table contains each of the elements only once. Assume for a moment that $4+a = 2$ meaning that $a = 3$. If $4+b$ is also equal to 2, b must also equal 3 because the sum 2 only occurs once in the row headed by 4. In the Z_6 table, on the other hand, some elements are repeated while others do not appear at all. If the products $4\cdot a$ and $4\cdot b$ both equal 2 for example, a is not necessarily the same as b. The product 2 appears twice in the fourth row so a could equal 2 since $4\cdot 2 = 2$ and b could equal 5 since $4\cdot 5 = 2$ also. Although the products $4\cdot 2$ and $4\cdot 5$ are the same, 2 does not equal 5. In $(Z_6\setminus\{0\},\cdot)$ if $4\cdot a = 4\cdot b$, a is not necessarily equal to b.

The term cancellation law is used to describe the situation that exists in $(Z_5,+)$ but not in $(Z_6\setminus\{0\},\cdot)$. Any commutative operational system $(S,*)$ has a cancellation law when for all a, b, and c in S

if $a * c = b * c$, then $a = b$.

The existence of a cancellation law can be seen most easily from the table for the operational system. If each element in the set appears once and only once in each row and column, then the operational system has a cancellation law.

In problem 4 in the previous exercises you listed all the groups and non-groups you have studied so far. Your list probably looked something like this:

Groups	Commutative?	Non-Groups
$(Z_n,+)$	yes	$(Z_n\setminus\{0\},\cdot)$, if n is composite
$(Z_n\setminus\{0\},\cdot)$, if n is prime	yes	(P,∇)
	___	(F, f)
(E, d)	yes	$(R, @)$
(G, d)	yes	$(W,+)$
$(A, \#)$	yes	$(W\setminus\{0\},\cdot)$

Which of these systems has a cancellation law? Begin with the groups. Check each table. If each element occurs only once in each row and column, then the system has a cancellation law. You should find that all of the groups listed have a cancellation law. In fact, all groups have a cancellation law. Now study the tables for each non-group. Of those listed, $(R, @)$, $(W,+)$ and $(W\setminus\{0\},\cdot)$ have a cancellation law, while (P,∇), (F,f) and those $(Z_n\setminus\{0\},\cdot)$ when n is composite do not. Therefore some non-groups have a cancellation law, others do not. However, all groups have a cancellation law.

It is very important that you do not include cancellation laws among the group properties. A cancellaton law is a characteristic of all groups and not a requirement. To be a group, an operational system must meet only three requirements. It must be associative, have an identity element and the inverse property. The cancellation law is a consequence of a system being a group. It is possible to prove that a cancellation law exists for all groups by using the three group properties. To do this assume that the first part of the law, $a * b = a * c$ is true and transform it so that $b = c$.

Proof

$$\begin{gathered} a * b = a * c \\ a^{I} * (a * b) = a^{I} * (a * c) \\ (a^{I} * a) * b = (a^{I} * a) * c \\ e * b = e * c \\ b = c \end{gathered}$$

Step 1 is the first part of the cancellation law. In step 2, a^{I} is introduced on both sides of the equal sign. Because the system is a group you know that a^{I} exists for all a in the set. Step 3 applies the associative property. Step 4 uses the inverse property, $a^{I} * a = e$. Step 5 involves the definition of the identity element, $e * b = b$. The proof is complete. You began with the fact that $a * b = a * c$ and showed by using all of the group properties that b must equal c.

Section 3. Non-Commutative Groups

Look again at the list of groups and non-groups. Notice that all of the groups you have studied so far are commutative. It would be very logical to ask if any non-commutative groups exist. The answer is yes. One of the simplest such groups involves a set that is not made up of numbers, and a new binary operation you have never met before.

Consider rearranging three books, labelled A, B and C on a shelf. There are several ways to order the books, besides A, B, C, for example, B, A, C or C, B, A. In all there are six possible ways to arrange three books. These six elements make up the set P. Each ordering is expressed in an array, for example,

$$\begin{vmatrix} A & B & C \\ C & A & B \end{vmatrix}$$

The top row is always the same A, B, C. It represents the original arrangement of the books. The bottom row shows the present arrangement of the books, in this case C, A, B. As you can see a list of the elements in P would be very cumbersome. To avoid this situation each arrangement is assigned a letter from e to j, thus P = {e, f, g, h, i, j}. The six members of P and their respective arrays are listed below:

$$e = \begin{vmatrix} A & B & C \\ A & B & C \end{vmatrix} \qquad f = \begin{vmatrix} A & B & C \\ B & C & A \end{vmatrix} \qquad g = \begin{vmatrix} A & B & C \\ B & A & C \end{vmatrix}$$

$$h = \begin{vmatrix} A & B & C \\ C & B & A \end{vmatrix} \qquad i = \begin{vmatrix} A & B & C \\ A & C & B \end{vmatrix} \qquad j = \begin{vmatrix} A & B & C \\ C & A & B \end{vmatrix}$$

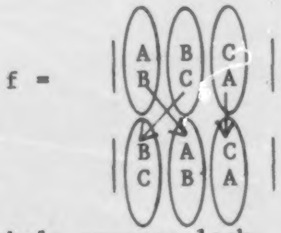
A binary operation must now be defined on the set P. The operation is called composition and is denoted by a small circle, $\circ$. For example the composition of f and g is written $f \circ g$ and is read f "follows" g. Pay special attention to the order in which the elements are combined. The phrase f "follows" g indicates that you begin with g, then f "follows" it. This is exactly opposite of other binary operations you are familiar with. So remember, $f \circ g$ means g is first, then f. You may find this confusing at first but composition is a very common binary operation and you will use it many times both in transformational geometry and in later courses.

The next step is to show that P and the binary operation of composition form an operational system. Every pair of elements in P must have an image also in P. To find the image of, say $f \circ g$, begin by replacing f and g with their respective arrays.

$$f \circ g$$

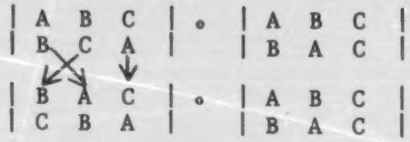
$$\begin{vmatrix} A & B & C \\ B & C & A \end{vmatrix} \circ \begin{vmatrix} A & B & C \\ B & A & C \end{vmatrix}$$

In g the order of the books is changed from A, B, C to B, A, C. The element f must change B, A, C to another arrangement, and herein lies the problem. Like all elements of P, f begins with the original arrangement A, B, C. Therefore f will have to be rearranged so the top row is B, A, C. To do this move the books in pairs as shown below.



Now f can follow g. The whole process looks like this.

$f \circ g$



So what began as A, B, C was changed to B, A, C in g, then to C, B, A in f. Therefore, in the end the order A, B, C was changed to C, B, A or

$$\begin{vmatrix} A & B & C \\ C & B & A \end{vmatrix}$$

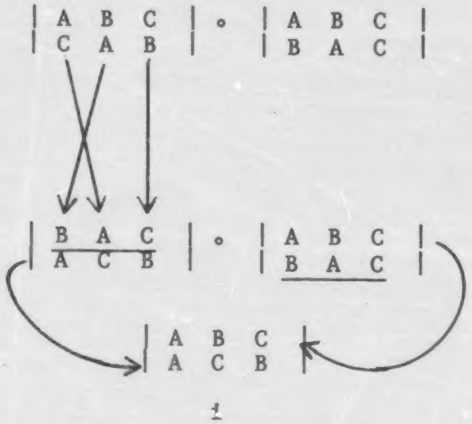
This is the arrangement h, so $f \circ g = h$.

The next three examples are very important. Study each one carefully. Read all the little notes. They should help clarify the process.

Ex. 1

$j \circ g$

Rearrange j.
The 2 underlined rows must be the same.



start

final result

Ex. 2

$g \circ j$

$$\begin{vmatrix} A & B & C \\ B & A & C \end{vmatrix} \circ \begin{vmatrix} A & B & C \\ C & A & B \end{vmatrix}$$

$$\begin{vmatrix} \underline{C \quad A \quad B} \\ C \quad B \quad A \end{vmatrix} \circ \begin{vmatrix} A \quad B \quad C \\ \underline{C \quad A \quad B} \end{vmatrix}$$

$$\begin{vmatrix} A & B & C \\ C & B & A \end{vmatrix}$$

$$h$$

Ex. 3

$$i \circ i$$

$$\begin{vmatrix} A & B & C \\ A & C & B \end{vmatrix} \circ \begin{vmatrix} A & B & C \\ A & C & B \end{vmatrix}$$

$$\begin{vmatrix} A & C & B \\ A & B & C \end{vmatrix} \circ \begin{vmatrix} A & B & C \\ A & C & B \end{vmatrix}$$

$$\begin{vmatrix} A & B & C \\ A & B & C \end{vmatrix}$$

$$e$$

Look closely at examples 1 and 2. In example 1 $j \circ g = i$, while in example 2, $g \circ j = h$. Therefore, $j \circ g \neq g \circ j$, and "follows" is not a commutative operation on the set P.

Keep this in mind as you complete the table outlined in the exercises below.

EXERCISES

1. Complete the table for (P,∘). The entry h is $f \circ g$.

∘	e	f	g	h	i	j
e						
f			h			
g						
h						
i						
j						

2. Is (P,∘) an operational system? Why?
3. Is there an identity element? If so name it. Remember the identity element should leave the order of the items unchanged.
4. Does this system have the inverse property? Name the inverse of each element in the set if one exists.
5. Does the associative property hold, that is, does

   $h \circ (g \circ j) = (h \circ g) \circ j$?

   In the right half perform $h \circ g$ first, and then perform this result after doing j.

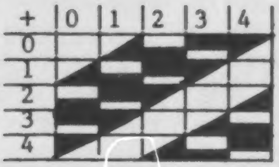
6. Check the following examples of associativity.

$$g\circ(h\circ i) = (g\circ h)\circ i$$
$$j\circ(g\circ i) = (j\circ g)\circ i$$
$$f\circ(h\circ j) = (f\circ h)\circ j$$

7. Is $(P,\circ)$ a group?
8. Is $(P,\circ)$ a commutative group?
9. Does $(P,\circ)$ have a cancellation law? Why?

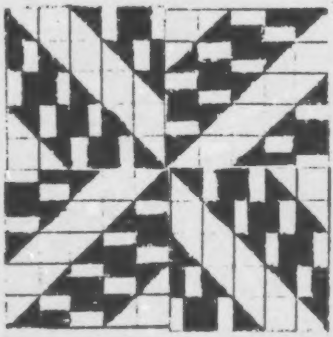
Section 4. Designs from Finite Systems

In this section you will have the opportunity to apply your knowledge of finite systems to geometric designs. To create your design choose any finite system and make a table using geometric shapes instead of numbers. For example, assume you choose $(Z_5,+)$. Instead of putting a number in each square of the table, put in a different geometric shape for each element. Zero might be left blank, wherever it appears in the table. One might appear as ◪. Since 4 is its additive inverse it should be colored ◩ in the opposite manner. Two and three are also inverses so 2 might be represented as ⬓ and 3 as ⬒ . The complete table appears below.

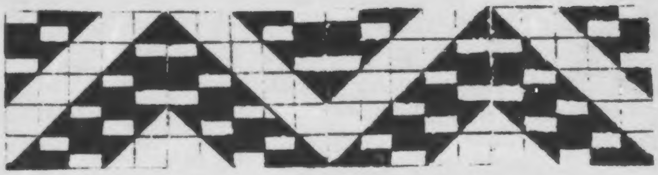


The symmetry of the table is immediately obvious as are commutativity and the inverse property.

By making several tables and cutting them out, you can arrange them in many interesting ways. One possibility is show below. Four tables were combined by rotating each 90 degrees.



By making additional tables that are mirror images of the one shown, this zig-zag pattern can be formed.



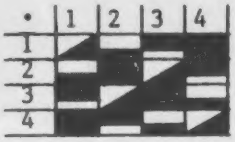
Multiplication tables have very different patterns. In this table for $(Z_5\setminus\{0\},\cdot)$ inverses are again colored the opposite of one another.

1 = ◩

2 = ⬓

3 = ⬒

4 = ■



These systems too can be used in multiples to form very interesting designs. The larger systems often make a fine showing used singly or in pairs.

EXERCISES

Assign a shape to each element of these systems.
Make a table on graph paper using the shapes.

1. $(Z_6,+)$ 2. $(Z_6\setminus\{0\},\cdot)$ 3. $(Z_7,+)$ 4. $(Z_7\setminus\{0\},\cdot)$.

5. Make a design using any Z_n you wish. Color your design with colored pencils or felt tip markers.

6. Addition tables have a very different appearance from multiplication tables. Explain why.

REVIEW OTHER GROUPS

1. Terms to define: Group (Use (S,*))
   Cancellation Law

2. Give an example of a commutative group.

3. Give an example of a non-commutative group.

4. Determine whether or not this operational system is a group. Is the system commutative? Does it have a cancellation law?

Ω	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	a	f	d	e
c	c	a	b	e	f	d
d	d	e	f	a	b	c
e	e	f	d	c	a	b
f	f	d	e	b	c	a

Answers

1. See text.

2. See page 3-6.

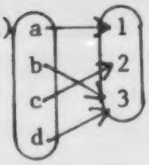
3. (P,∘)

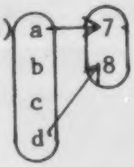
4. The system has an identity element, a, and each element has an inverse, b and c are inverses and all other elements are their own inverses. The system is associative and therefore a group. Like all groups the system has a cancellation law but it is not commutative.

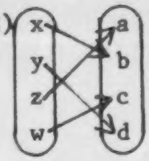
CUMULATIVE REVIEW CHAPTERS 1-3

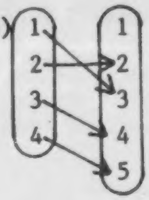
1. Study each arrow diagram carefully, then answer the questions.

   1) Is it a mapping?
   2) What is the domain?
   3) What is the codomain?
   4) What is the range?
   5) Is the mapping one-to-one?
   6) Is the mapping onto?

A)


B)


C)


D)


2. Make arrow diagrams, using whole numbers, for each rule of assignment.

   a) $n \to 2n + 4$. b) $n \to n^2 + 1$.

3. Indicate the restricted domain which will make these rules of assignment mappings from W to W.

   a) $n \to \frac{n-2}{5}$ b) $n \to \frac{n}{5} - 2$

4. Which $(Z_n, +)$ and $(Z_n \setminus \{0\}, \cdot)$ are groups?

5. Why is zero restricted in multiplication and division?

6. Which $(Z_n \setminus \{0\}, \cdot)$ do not have a cancellation law? Explain why.

7. Is subtraction a binary operation on any Z_n? Explain.

8. Is division a binary operation on any $(Z_n \setminus \{0\}, \div)$? Explain.

9. Determine if the system below is a group. Answer these questions.
   - a) Is there an identity element? Identify it.
   - b) Is there an inverse for every element? Name them.
   - c) Is the system associative? Give one example.
   - d) Is the system a group?
   - e) Is the system commutative?
   - f) Is there a cancellation law?

□	a	b	c	d	e
a	c	d	e	a	b
b	d	e	a	b	c
c	e	a	b	c	d
d	a	b	c	d	e
e	b	c	d	e	a

10. Which properties are evident directly from the table? How is each property determined?

11. Which group property is not evident directly from the table? How do you determine if the system has this property?

12. Simplify these expressions in the operational system indicated.

    a) $4 + ((-3) + (-5))$ in $(Z_7,+)$

    b) $4^{-1} \cdot 3^{-1}$ in $(Z_7 \setminus \{0\},\cdot)$

    c) $5^{-1} \cdot (2 \cdot 3^{-1})$ in $(Z_8 \setminus \{0\},\cdot)$

    d) $4 \div 6$ in $(Z_7 \setminus \{0\},\div)$

    e) $(2 \div 3) \div 4$ in $(Z_5 \setminus \{0\},\div)$

    f) $3 - 7$ in $(Z_8,-)$

Answers

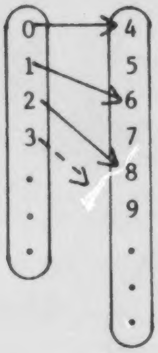
1. A. 1)Yes, 2) {a,b,c,d}, 3) {1,2,3}, 4) {1,2,3}, 5)No, 6)Yes.

   B. 1)No.

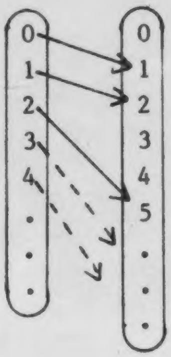
   C. 1)Yes, 2){x,y,z,w}, 3){a,b,c,d}, 4){a,b,c,d}, 5)Yes, 6) Yes.

   D. 1)Yes, 2){1,2,3,4}, 3){1,2,3,4,5}, 4){2,3,4,5}, 5)Yes, 6)No.

2. a)



b)



3. a) {7, 12, 17, ...}, b){10, 15, 20, ...}

4. All $(Z_n,+)$ are groups; $(Z_n \setminus \{0\},\cdot)$ are groups only when n is prime.

5. Zero has no multiplicative inverse.

6. All systems where n is composite.

7. Yes, all $(Z_n,-)$ are operational systems.

8. Division is a binary operation on Z_n only when n is prime.

9. a)Yes, d.
   b)Yes, a and b are inverses, c and e are inverses, d is its own inverse.
   c)Yes, $a \square (c \square e) = (a \square c) \square e$

$$a \square d = e \square e$$
$$a = a$$

   d)Yes.
   e)Yes.
   f)Yes.

10. The identity element, inverse property and commutative property are evident directly from the table. To locate the identity element find a row and column

element. If this element occurs in every row and column the system has the inverse property. If the two sides of the diagonal are the same, the system is commutative. In addition, if each element in the set occurs only once in each row and column, the system has a cancellation law.

11. To determine if the system is associative, every possible example must be checked.

12. a) 3, b) 3, c) 6, d) 3, e) 1, f) 4.

PRE-ALGEBRA GROUPS

Chapter 4

Integers

Chapter 4 Integers

Section 1. The Set of Integers

Now it is time to return to the set of whole numbers. You may recall that the operational system (W,+) is not a group. The system lacks the inverse property. For example, what whole number can be added to 4 to give the additive identity element 0? In other words, what is the solution of 4 + ____ = 0? There is no whole number solution for this equation; 4, like all whole numbers except 0, has no additive inverse.

The only additive groups you have studied are those involving finite sets, Z_n. Each element, a, has an inverse, written - a. Using this same symbol a new set of numbers can be formed. These are the additive inverses of the whole numbers. They are also called negative numbers. Thus, -6 is read either as the additive inverse of 6 or as negative 6. In some textbooks negative numbers are written with a short raised bar, for example $^{-}6$. This is equivalent to -6; that is $-6 = {}^{-}6$. In this unit negative numbers will be written using the simple negative bar in its normal location, hence -6, not $^{-}6$. The complete set of negative numbers is {-1, -2, -3, -4, ...}. It contains the additive inverses of every whole number except 0, which is its own additive inverse. The set formed by combining the set of whole numbers and the set of inverses is called the set of integers and is designated by the capital letter Z, where

$$Z = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}.$$

In order to determine if (Z,+) is a commutative group, you need to learn how to add integers. This topic is covered in your textbook, so study it carefully and do many practice problems before continuing.

Section 2. (Z,+) A Commutative Group

In the exercises just completed you discovered that every ordered pair of integers has an image in the set of integers. Because this mapping exists, (Z, +) is an operational system. It remains to determine if (Z, +) is a commutative group. The system is associative. For all integers a, b, and c

$$(a + b) + c = a + (b + c)$$

For instance,

$$\begin{aligned} (3 + -5) + -7 &= 3 + (-5 + -7) \\ -2 + -7 &= 3 + -12 \\ -9 &= -9 \end{aligned}$$

As in all additive systems, zero is the identity element. For every integer a,

$$a + 0 = a \text{ and } 0 + a = a.$$

For example, $3 + 0 = 3$ and $0 + 3 = 3$ or $-8 + 0 = -8$ and $0 + -8 = -8$. And the inverse property is finally satisfied.

For every integer a

$$a + (-a) = 0 \text{ and } (-a) + a = 0.$$

For example, if $a = 5$, then $-a = -5$ which gives

$$5 + -5 = 0 \text{ and } -5 + 5 = 0.$$

If $a = -7$, then $-a = -(-7) = 7$ which gives

$$-7 + 7 = 0 \text{ and } 7 + -7 = 0.$$

Since $a + b = b + a$ for all a and b in the set, (Z, +) is also commutative and therefore forms a commutative group.

Section 3. Subtraction

Very early in this unit subtraction on the set of whole numbers was found to be a restricted operation. The ordered pairs (a, b) had images in W only when a was greater than or equal to b; that is, for a, b and c in W

$$(a, b) \rightarrow c \text{ only if } a \geq b.$$

Using the set of integers it may be possible to remove this restriction. You first need to learn to subtract integers. This topic can be found in your own textbook.

In the exercises just completed, you should have found that

$$a - b = a + (-b), \text{ for all a and b in Z.}$$

Since every subtraction problem can be rewritten as an addition problem, and (Z,+) is an operational system, (Z, -) is also an operational system.

To determine if (Z, -) is a group, begin with the associative property. Below is one example:

Ex. 1

$$4 - (6 - 8) = (4 - 6) - 8$$
$$4 - -2 = -2 - 8$$
$$6 = -10$$

But $6 \neq -10$, therefore, although (Z, -) is an operational system, it is not a group. The system also has no identity element. Although zero might at first appear to be the identity element because $a-0 = a$, the reverse is not true, $0-a \neq a$. So there is no identity element for subtraction and therefore no inverse property. Is subtraction commutative? No, $5-2 \neq 2 - 5$.

Notice that subtraction of integers exactly parallels subtraction for all finite numbers systems. Although you will soon add elements to the set of integers forming a new larger set, subtraction will continue to lack all group properties.

Section 4. Multiplication and Division

The next binary operation to be considered is multiplication. You know that $(W\setminus\{0\},\cdot)$ is an operational system. It is reasonable to expect that $(Z\setminus\{0\},\cdot)$ is also an operational system, because the set of integers is formed from the set of whole numbers. However, to simplify expressions such as $4 \cdot -3$ or $-5 \cdot -6$ requires that you learn some new techniques. These can be found in your own textbook.

As you practiced multiplying integers you discovered that all pairs of integers have a unique product indicating further that $(Z\setminus\{0\},\cdot)$ is an operational system. To determine if the system is a group, begin, as usual, with the associative property. Since $(W\setminus\{0\},\cdot)$ is associative, a few examples should suffice to establish that $(Z\setminus\{0\},\cdot)$ is also associative.

Ex. 2

$$(-3\cdot 4)\cdot -2 = -3\cdot(4\cdot -2)$$
$$-12\cdot -2 = -3\cdot -8$$
$$24 = 24$$

Ex. 3 $(-5\cdot-4)\cdot-2 = -5\cdot(-4\cdot-2)$

$20\cdot-2 = -5\cdot 8$

$-40 = -40$

One is the identity element for integer multiplication just as it is for all multiplicative systems.

Ex. 4 $-2\cdot 1 = -2$ Ex. 5 $1\cdot -5 = -5$

Recall that the set of whole numbers does not satisfy the inverse property. For example, $4\cdot\_\_\_ = 1$ has no solution in W. Is there any integer that correctly completes this same sentence? Unfortunately there is not. Therefore the set of integers has no inverse property for multiplication. This means, of course, that $(Z\setminus\{0\},\cdot)$ is not a group.

But this result is not at all surprising. Think of how the set of whole numbers is expanded to form the set of integers. The negative integers are the additive inverses of the whole numbers, not the multiplicative inverses. Neither $(W,+)$ nor $(W\setminus\{0\},\cdot)$ is a group because each lacks the inverse property. However, $(Z,+)$ is a group because the necessary inverses have been included. Only by enlarging the set of integers to include all multiplicative inverses will a group be possible for this binary operation.

This expanded set may also improve the situation for division. As yet, even in the set of integers such simple problems as $8 \div 3$ have no solution, meaning that division of integers is not an operational system.

In summary, addition of integers is a commutative group, multiplication of integers is a commutative operational system having all group properties except the inverse property; subtraction is also an operational system but has no group properties; and division of integers is not even an operational system.

The exercises that follow are designed to review your understanding of integers, before applying the rules for the order of operations in the next section.

EXERCISES

Perform the indicated operations across and down. To obtain the final entry, repeat the process using the numbers just computed. This will serve to check your work.

ADD

6	-3	
5	-7	

-8	-2	
7	-6	

9	-7	
-6	4	

8	-6	
-2	5	

8	-3	
5	-6	

SUBTRACT

-8	2	
7	-6	

-9	-3	
5	2	

7	8	
-3	-4	

-3	-6	
-8	-2	

7	-1	
-9	2	

MULTIPLY

7	-3	
6	-5	

-3	4	
-6	0	

-7	9	
2	-3	

-3	-7	
2	6	

2	-9	
3	-6	

DIVIDE

-12	6	
-4	2	

-48	-16	
12	4	

-60	-30	
-20	-10	

72	-36	
24	-12	

72	18	
-12	-3	

Time Test - Mixed Operations

-9+8 =

3 - 5 =

(-8) (7) =

-4 + -6 =

36 ÷ 9 =

(-5) (-6) =

-2 - 7 =

-42 ÷ 7 =

-4 - -9 =

-6 + -8 =

(-6) (-8) =

2 - 9 =

-45 ÷ -5 =

1 + -3 =

0 + -9 =

(-4) (2) =

-5 + -4 =

6•6 =

-4 - 7 =

3 + -9 =

-24 ÷ 3 =

5 - 9 =

72 ÷ -8 =

3 - 6 =

-6 - 5 =

-32 ÷ 4 =

-2 + 8 =

(-7) (7) =

-4 + -7 =

8 - 9 =

-81 ÷ -9 =

-3 + -5 =

-8 + 2 =

(-1) (8) =

3 + -4 =

5 - -4 =

48 ÷ -6 =

(-4) (7) =

-3 - 8 =

9 + -4 =

(-9) (8) =

-49 ÷ -7 =

-1 - 6 =

-30 ÷ 6 =

-6 + 5 =

2 - -8 =

-56 ÷ -8 =

-1 + 7 =

0 - 6 =

(-7) (-5) =

63 ÷ -9 =

8 + -6 =

-7 + -8 =

(-8) (4) =

-1 + 0 =

64 ÷ 8 =

-5 - 9 =

(-3) (-9) =

-35 ÷ -5 =

7 + -7 =

1 - 8 =

(-7) (-8) =

(-8) (-6) =

-7 - 8 =

-25 ÷ -5 =

1 + -9 =

-5 - -4 =

(-5) (-9) =

-9 + 7 =

-72 ÷ -9 =

6 - -8 =

(-9) (9) =

Number Completed ______________ Number Correct ______________

Product - Sum - Difference - Quotient

Complete the table. Subtraction is A - B, division is A ÷ B. All results are integers.

	÷	X	A	B	+	-
1.			-8	2		
2.		-18	6			
3.				-4	-12	
4.		-24			10	
5.				3		-18
6.		9			-10	
7.	4		-8			
8.			10	-5		
9.		-28		-2		

10.			10		8	
11.	-3		-18			
12.	1				-6	
13.					-9	-15
14.				4		-16
15.		28		-2		
16.		32			-12	
17.		32				-14
18.			-12			-18

Section 5. Order of Operations

There are four rules governing the order in which operations are performed on the set of whole numbers.

1. Clear all parentheses first using the rules that follow
2. Evaluate all exponents
3. Multiply and/or divide in order left to right
4. Add and/or subtract in order left to right

Before applying these rules to integers study the following two examples as a review.

Ex. 1.

$7 \cdot 8 - (4 + 3 \cdot 5) + 6 \cdot (5-2)$
$7 \cdot 8 - (4 + 15) + 6 \cdot (3)$
$56 - 19 + 18$
55

Ex. 2.

$$\frac{5 + 16}{14\div2} - \frac{3\cdot(8-4)}{6}$$

$$\frac{5 + 16}{14\div2} - \frac{3\cdot4}{6}$$

$$\frac{21}{7} - \frac{12}{6}$$

$$3 - 2$$

$$1$$

The order of operations is exactly the same for integers as it is for whole numbers. You will need to write out several steps for each expression. Be especially careful of signs.

Ex. 3

$48 + -12\cdot2 + 4\cdot -3$

$48 + -24 + -12$

12

Ex. 4

$-3\cdot(-5 - -6) + 4\cdot-7\cdot-2$

$-3\cdot(1) + 4\cdot-7\cdot-2$

$-3 + 56$

53

EXERCISES

1. $-36 -8\div4\cdot2 - -6=$
2. $15-24\div(-2\cdot6) - (-3)=$
3. $[144\div(12 -(-4))]\cdot(8\div-2)=$
4. $(-100 -(-25))\div -5+15\cdot5=$
5. $-100-25\div5 -(-15)\cdot5=$
6. $-18-6\div-3\cdot2 +(-8)\cdot1=$
7. $18 +(-6)\div-3\cdot2-8\cdot-1=$
8. $-18 -(-6)\div3\cdot(-2)+8\div8=$
9. $-7+4\cdot(-2)-6-(8-9)=$
10. $-7 -(-4)(-2) + (-6) +(-8)+9=$
11. $-2\cdot(-7+4)-6-(8-9)=$
12. $-8\cdot(-6)\cdot(-2)+3\cdot(-8)\div1=$
13. $5\cdot(8(-6)+3) -(-9)\div3=$
14. $-5\cdot(-8\cdot6 -(-3))+9\div-3=$

26. $8\cdot(-6) + 3\cdot(-5)=$
27. $-72\div(-18)-4+31\cdot(-7)=$
28. $(-31\cdot7+73)\div18-4=$
29. $(-13\cdot8 -(-11))\div31=$
30. $31 +(-2)(13(-8)+11)=$
31. $-11-(-13\cdot8 -(-7))=$
32. $14+3\cdot(6-24)\div2=$
33. $-3\cdot(6-24)\div9-8=$
34. $-5-(18-31)\div13-5=$
35. $-5\cdot[-6+14\cdot(-17+13)\div4]=$
36. $-3-8\cdot(4-7)-6+3=$
37. $(-17+3)\div7(-6)+8-2=$
38. $4-7+3\div(-3)(-11)=$
39. $-37+4\cdot6-28\div7-9=$

15. $5\cdot(-12+4) + -8\div4\cdot2=$

16. $5\cdot-12+4 + -8\div4\cdot2=$

17. $4\cdot-3+21\div3+4=$

18. $-1\cdot(-2-3\cdot-4)-5\cdot(-6 - -7)=$

19. $\dfrac{-1 - -2\cdot3\cdot-4}{-5} =$

20. $-3\cdot-4\cdot-5 -(6\cdot-7 - -8\cdot-9)=$

21. $-1-2\cdot-3\cdot-4 - -5\cdot-6-7\cdot-8=$

22. $4-(3+-8)+4\div-2=$

23. $\dfrac{19+3\cdot-6-4\cdot-2-3\cdot(2-5)}{-2\cdot-4 - -3\div3} =$

24. $\dfrac{-18-3}{-3} - \dfrac{15-3}{3-15} =$

25. $4\cdot-3 - -2\cdot(5-7\cdot-2)-6\cdot(7\cdot3-12)=$

40. $\dfrac{-1-(-2 - -3\cdot-4) - -5}{-(6 - -7) - -8 -1}=$

41. $1 - -2+3\cdot[4-(5-6)+7]=$

42. $4\cdot-3+21\div(3+4)=$

43. $-1 - -2-3\cdot-4 - -5\cdot-6 - -7=$

44. $\dfrac{-1\cdot-2\cdot(3 - -4)}{6-7 - -8} =$

45. $8\cdot -2 + 4\cdot(2-7)-(5-9)=$

46. $\dfrac{8-3\cdot(-4-7)-2-5}{-5 - -3} =$

47. $\dfrac{-6\cdot5\cdot-4\cdot3\cdot-2\cdot-1}{-6\cdot3\cdot-2} =$

48. $-2\cdot[3-(4 - -3\cdot5)+2\cdot-6] + -4\cdot7 =$

49. $-2\cdot4\cdot-6 + -2+4 + -6 - -2-4 - -6=$

Section 6. Order of Operations - Exponents

Expressions involving exponents require a little more explanation. You already know that

$$3^4 = 3\cdot3\cdot3\cdot3 = 81$$

If instead you want to raise -3 to the fourth power, that is, multiply -3 by itself four times, write

$$(-3)^4 = -3\cdot-3\cdot-3\cdot-3 = 81$$

Likewise,

$$(-2)^3 = -2\cdot-2\cdot-2 = -8$$
$$(-5)^2 = -5\cdot-5 = 25$$

The parentheses are necessary in this case. Without them the meaning is entirely different.

The expression, -3^4, is the additive inverse of 3^4, that is

$$-3^4 = -(3\cdot3\cdot3\cdot3) = -81.$$

Likewise, -2^3 is the additive inverse of 2^3, that is

$$-2^3 = -(2 \cdot 2 \cdot 2) = -8.$$

Study these examples carefully:

Ex. 1 a) $(-4)^3 = -4 \cdot -4 \cdot -4 = -64$
b) $(-7)^2 = -7 \cdot -7 = 49$

Ex. 2 a) $-4^3 = -(4 \cdot 4 \cdot 4) = -64$
b) $-7^2 = -(7 \cdot 7) = -49$

Notice that in example 1, $(-4)^3 = -64$ while $(-7)^2 = 49$. In one case a negative number raised to a power yields a negative result, while in the other, a negative number raised to a power gives a positive answer. In the exercises you will develop a rule for how the value of the exponent determines the sign of the answer. In example 2, the fact that $-4^3 = -64$ and $-7^2 = -49$ implies that the additive inverse of a whole number raised to a power is always negative. You will have an opportunity to test this conclusion later.

Another possible expression you may need to simplify is $-(-2)^3$. This expression is read the additive inverse of $(-2)^3$. Since $(-2)^3 = -2 \cdot -2 \cdot -2 = -8$, and the additive inverse of -8 is 8, then $-(-2)^3 = 8$. The total process is detailed below.

$$\begin{aligned} -(-2)^3 &= -(-2 \cdot -2 \cdot -2) \\ &= -(-8) \\ &= 8 \end{aligned}$$

Ex. 3

$$\begin{aligned} -(-3)^4 &= -(-3 \cdot -3 \cdot -3 \cdot -3) \\ &= -(81) \\ &= -81 \end{aligned}$$

Ex. 4

$$\begin{aligned} -(-5)^3 &= -(-5 \cdot -5 \cdot -5) \\ &= -(-125) \\ &= 125 \end{aligned}$$

These new facts about exponents are needed to simplify some expressions.

Ex. 5

$(-2)^5 - 3 \cdot -6$

$(-2 \cdot -2 \cdot -2 \cdot -2 \cdot -2)\ -3 \cdot -6$

$-32 - -18$

-14

Ex. 6

$5^2 - (-2 \cdot 3)^2$

$5^2 - (-6)^2$

$25 - 36$

-11

Ex. 7

$$\frac{2^2 \cdot -3}{-6} + \frac{-(-6)^2}{-2^2 \cdot 3}$$

$$\frac{4 \cdot -3}{-6} + \frac{-(36)}{-4 \cdot 3}$$

$$\frac{-12}{-6} + \frac{-36}{-12}$$

$$2 + 3$$

$$5$$

EXERCISES

Simplify each expression.

1. $(-3)^5$ 2. -4^3 3. -4^4 4. -2^5 5. $(-2)^5$
6. 7^3 7. -7^3 8. $(-7)^3$ 9. $-(-7)^3$ 10. $-(-3)^2$

Simplify each expression. Look for a pattern.
11. $(-2)^2$ 12. $(-2)^3$ 13. $(-2)^4$ 14. $(-2)^5$ 15. $(-2)^6$ 16. $(-2)^7$

17. Look at problems 11 through 16. For which exponents are the final values negative? For which exponents are the final values positive? Complete the following rule:

$(-2)^n$ is positive if n is _______ and
$(-2)^n$ is negative if n is _______.
Does this rule hold for other negative numbers besides -2?

Simplify these expressions. Again look for a pattern.

18. -3^2 19. -3^3 20. -3^4 21. -3^5 22. -3^6.

23. Write a rule based on the results of problems 18-22.

Simplify each expression. Be very careful.

24. $-2\cdot(-3)^3=$

25. $5\cdot3^2 - 2^3=$

26. $-2^3\cdot7 + 3^3=$

27. $3\cdot2 + 2^3\cdot(-2)^2=$

28. $-(5-2^3)^2=$

29. $-(5-2^3)^3=$

30. $(-3)^2\cdot(2^2\cdot-6 +7^2)=$

31. $[(-2)^3 - -2\cdot-3]^2=$

32. $(1-2)\cdot(3^2 + -4\cdot-5)=$

33. $(-2)^5 + -3\cdot4\cdot-5+(-6)^2 - -7\cdot-8=$

34. $(-2)^3\cdot(-3)^3-(-4)^3+5^3-6+(-7)^2 + -8 + -9=$

35. $-6\cdot5\cdot-4\cdot3\cdot-2\cdot1+4\cdot3^3+2=$

36. $\dfrac{-9-(3-8)}{(2-5)^2} \cdot \dfrac{(-3)^4}{3^2} =$

37. $\dfrac{2\cdot-4\cdot-6}{-2\cdot4} - \dfrac{3\cdot5\cdot6}{(-3)^2} + \dfrac{4\cdot-7}{(-2)^2}$

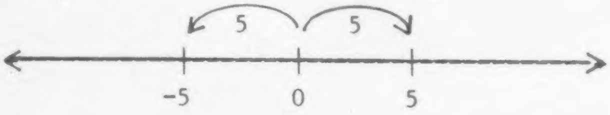
Section 7. Absolute Value

A number line is a familiar idea to most students. You are probably aware that every integer can be assigned to a point on a line. The integer is called the coordinate of the point. Because of what you have learned in Chapter 1, you should recognize this situation as a mapping from the set of integers to the set of points on the line. Since there are many points on the line that are not images of any integer, the mapping is not onto. It is a one-to-one mapping, however. Each point serves as the image of one and only one integer.

To construct a number line it is only necessary to assign zero and one to two different points on a line.

All other integers are assigned to points on the line based on the distance between zero and one. For example to locate 2, measure the distance from 0 to 1 and mark off that distance to the right of one.

All positive integers are located at equal intervals to the right of zero. The negative integers are found to the left of zero. Negative two for example is the same distance to the left of zero as positive two is to the right. Likewise 5 and -5 are the same distance on opposite sides of zero. Every integer and its additive inverse are the same distance on opposite sides of zero. Since it would not be correct to say that 5 = -5, some symbol is needed to indicate that these two integers are equal distances from 0. Two straight lines are used, one in front of the integer and one behind, for example |5| and |-5|. This concept is given the name absolute value. The absolute value of a number is the distance that number is from zero. The expression |-5| is read "the absolute value of -5" and is equal to 5 because -5 is 5 units from 0. The absolute value of 5, written |5|, is also equal to 5. On the number line,



|-5| =5 |5|= 5

Absolute value is also a mapping. What are the domain and the range? Since you can find the absolute value of any integer the domain is that set. But it is impossible to have a negative distance. The absolute value of a number is always positive or zero, so the range is the set of whole numbers. Keep this in mind as you study the following examples

Ex. 1 $|-3| = 3$

Ex. 2 $|7| = 7$

One of the many uses of absolute value involves finding the distance between two points. For example if you were asked to find the distance between 8 and 3, you would subtract. But two problems arise immediately. First, subtraction is not a commutative operation, so $3-8 \neq 8 - 3$. And second, distance must be positive so the correct expression would be 8-3 or 5 not 3-8 which equals -5. Notice however that the two results 5 and -5 differ only in sign, that is, they have the same absolute value. Expressing this in symbols,

$$\begin{aligned} |3 - 8| &= |8 - 3| \\ |-5| &= |\ 5| \\ 5 &= 5 \end{aligned}$$

Therefore, to find the distance between any two points on the number line, subtract their coordinates in either order, then take the absolute value of the difference.

Ex. 3. Find the distance between -8 and 3

$$\begin{array}{lll} |-8-3| & \text{or} & |3- -8| \\ |-11| & & |\ 11| \\ 11 & & 11 \end{array}$$

Ex. 4. Find the distance between -9 and -2

$$\begin{array}{lll} |-9- -2| & \text{or} & |-2- -9| \\ |-7| & & |\ 7\ | \\ 7 & & 7 \end{array}$$

Any operation can be performed on numbers in absolute value. If the operation is inside the absolute value signs as in the distance problems above, perform the operation first, then take the absolute value of the result.

Ex. 5
$$\begin{array}{l} |3+ -5| \\ |\ -2| \\ 2 \end{array}$$

Ex. 6
$$\begin{array}{l} |8\cdot -2| \\ |\ -16| \\ 16 \end{array}$$

If the operation is outside the absolute value signs, take the absolute value first, then perform the operation.

Ex. 7 $|3| + |-9|$
$3 + 9$
12

Ex. 8 $|-12| \cdot |-3|$
$12 \cdot 3$
36

To simplify expressions like $-|-6|$ begin be reading the expression, "the opposite of the absolute value of -6." So first take the absolute value of -6, then find the additive inverse of that result.

$-|-6|$
$-(6)$
-6

Study these additional examples

Ex. 9 $-|-7|$
-7

Ex. 10 $-|2-9|$
$-|-7|$
-7

Ex. 11 $-|-3 \cdot -4|$
$-|12|$
-12

Numbers in absolute value can be combined with others not in absolute value. Follow the general rules for order of operations.

Ex. 12 $3 \cdot |-2| - |4-7|$
$3 \cdot 2 - |-3|$
$6 - 3$
3

Ex. 13 $-2 \cdot |5 + -8| - |(-2)^3|$
$-2 \cdot |-3| - |-8|$
$-2 \cdot 3 - 8$
$-6 - 8$
-14

EXERCISES

Simplify

1. $|-9|$ 2. $|28|$ 3. $|2 \cdot -8|$ 4. $|6+3|$
5. $|-3 + -7|$ 6. $2 \cdot |-3|$ 7. $-|-9|$ 8. $|-3| \cdot |-6|$

Simplify. Look for a pattern in each pair of problems

9. $|3| + |6|$ 10. $|3+6|$ 11. $|2| + |-6|$ 12. $|2 + -6|$
13. $|-6| + |-9|$ 14. $|-6 + -9|$ 15. $|2| \cdot |3|$ 16. $|2 \cdot 3|$
17. $|2| \cdot |-7|$ 18. $|2 \cdot -7|$ 19. $|-6| \cdot |-8|$ 20. $|-6 \cdot -8|$

21. Study the problems involving addition. Is it true that $|a| + |b| = |a + b|$?

22. What relationship, if any, exists between $|a| + |b|$ and $|a + b|$?

23. Study the problems involving multiplication. Is it true that $|a| \cdot |b| = |a \cdot b|$?

Simplify each expression.

24. $5 \cdot |3-8| + |-6|$

25. $-3 \cdot |-3| + |-3|$

26. $-|-6| - |6|$

27. $2 \cdot |3 \cdot (4-6)|$

28. $|3-2 \cdot 5| - |-6-2|$

29. $\dfrac{|3 \cdot -4|}{2^2}$

30. $3 - |4 \cdot (2^2-7)|$

31. $6 \cdot |-3 \cdot (5+-9)|$

32. $|3| + |-3| - |7-2|$

33. $|7 \cdot 3| - |-3 \cdot 7|$

34. $-2 \cdot |-6| - |3 \cdot -6| + 2$

35. $-5 - |-5| + 7 \cdot 2 - 5$

36. $-2 \cdot |-3-7| - 3 \cdot |-6+4|$

37. $\dfrac{-3-|-4| + 2 \cdot |-6|}{|-4-1|}$

38. $\dfrac{6 \cdot -2 \cdot |-3| \cdot |4|}{|3| \cdot |-6|}$

39. $\dfrac{-7-2 \cdot |-7| + -6 - -3}{-2 \cdot |6+-2|}$

40. $-2 \cdot |-3| \cdot |-4| + |-3| + |-4|$

41. $\dfrac{-4 \cdot |-6-2| - |2-6|}{|3-5|}$

42. $\dfrac{-4 \cdot -3 \cdot |-2| + |5-6|}{|(-2)^2+1|}$

43. $(-3)^3 - |(-3)^3|$

44. $2^4 - |(-2)^3| - 2$

45. $(-2)^5 \cdot |-2+3| - |-6-22|$

46. $7 - |(-2)^2| + 3 \cdot |(-2)^3|$

47. $\dfrac{-6 \cdot |(-3)^2| - 5 \cdot 3^2 + (2 \cdot -3)^2}{(-3)^4 \div (-3)^2}$

REVIEW CHAPTER 4 INTEGERS

Discuss the set of integers and each of the four operations: addition, subtraction, multiplication and division. Explain which are operational systems. Explain which operations on the set of integers form commutative groups and which do not. Discuss each property in detail.

Simplify each expression.

1. $-31\cdot 4\cdot 2\div -8 + 9 =$

2. $|-9 -7\cdot -2| + |-3| =$

3. $-8\cdot 4 + 7\cdot 5 - 3\cdot -6 =$

4. $-6 + 7\cdot(3-8) -4 =$

5. $-3\cdot(-8+12) - 8 - (7-8) =$

6. $|-4+8-13| + 6\cdot -3 =$

7. $-4\cdot -9 - 6\cdot -4 - 7 + 12 =$

8. $|-9 - (5-9)| + |8\cdot -4| -6\cdot|-7| =$

9. $4\cdot[-5 - 6\cdot(7-9) + 5] + 4 =$

10. $-6\cdot|-8-4| -4\cdot|-5+8| =$

11. $[(-4)^3 - 9\cdot -6]^2 =$

12. $\dfrac{-8 - (-6- -4\cdot -3) - -4}{6 - 9- -11-1} =$

13. $\dfrac{12 - 4\cdot|(-2)^4| - 3\cdot 4^2}{|(-3)^2 + 8| + -4\cdot -2} =$

14. $-12\cdot(5-4\cdot 2+3)\cdot(-4)^3 + 16 =$

15. $(-5)^3 - (4-4^2) + |6-9| =$

ANSWERS 1. 40, 2. 8, 3. 21, 4. -45, 5. -19, 6. -9, 7. 65, 8. -5, 9. 52, 10. -84, 11. 100, 12. 2, 13.- 4, 14. 16, 15. -110

PRE-ALGEBRA GROUPS

Chapter 5

Transformational Geometry

Chapter 5 TRANSFORMATIONAL GEOMETRY

Section 1. Symmetry

Many objects in nature possess a certain balance or symmetry. The sketches in Fig. 1 illustrate one type of symmetry - line symmetry.

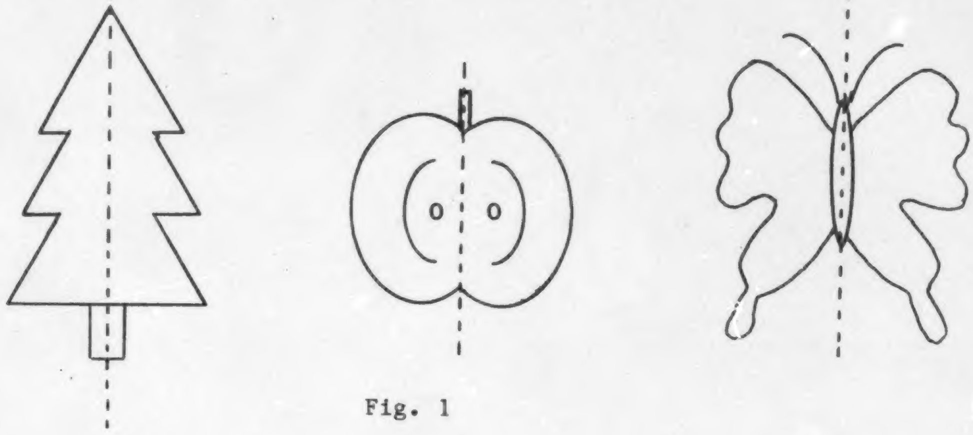


Fig. 1

An object has line symmetry if the part on one side of the line exactly matches the part on the other side. This type of symmetry is also very common in mathematics particularly among geometric figures. The rhombus, triangle, and trapezoid in Fig. 2 each have at least one line of symmetry.

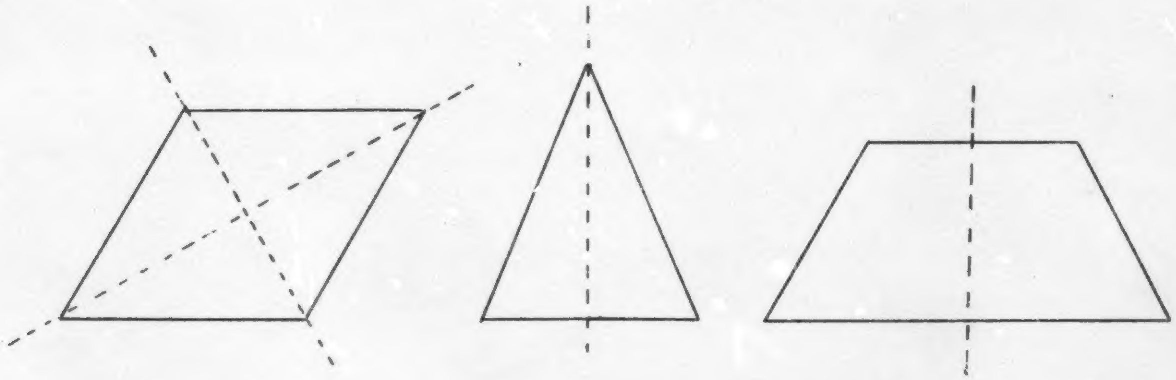


Fig. 2

Lines of symmetry can be determined by folding the figure so that the two halves match exactly. But since this is not always physically possible, a mirror may be used to locate the lines of symmetry. Position the mirror so that the reflection, together with the portion of the figure not covered by the mirror, look exactly like the original figure. Then, draw a line along the edge of the mirror. This line is one of the lines of symmetry of the figure. Since there may be more than one such line, always try to find another placement for the mirror.

Look carefully at the designs in Fig. 3:

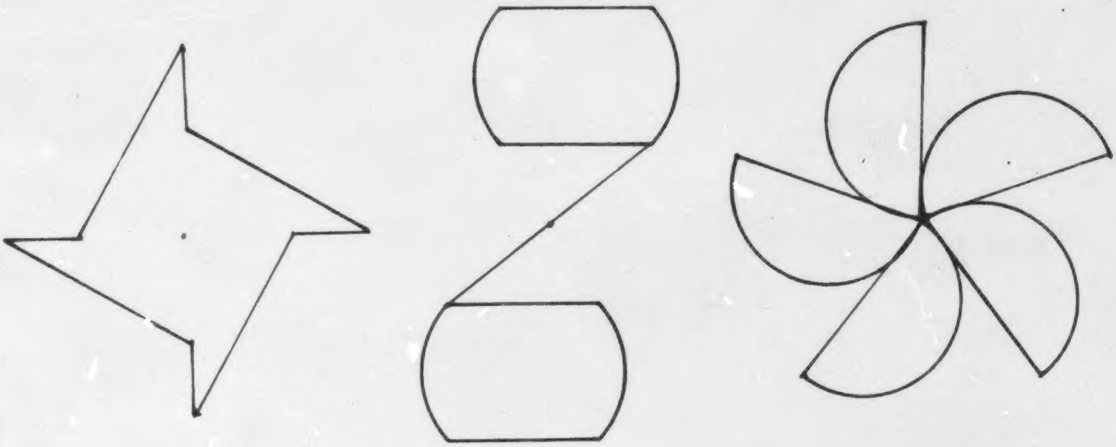


Fig. 3

Each design possesses a certain symmetry, but there are no lines of symmetry. If you make a tracing of the figure and anchor it on top of the original with a pin at the center, each figure will match itself when rotated a certain amount. These designs all have rotational symmetry; the first matches itself when rotated one-fourth turn, the second when rotated one-half turn, and the third when rotated one-fifth turn.

In the exercises you will meet many more examples of both rotational and line symmetry. These two important concepts will be studied in great detail in the following sections.

EXERCISES

1. Make a list of the capital letters of the alphabet that have one line of symmetry. Indicate the line on each letter.

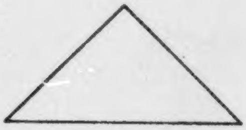
2. Make a list of the capital letters that have two lines of symmetry. Indicate the lines on each letter.

3. Do any letters have more than two lines of symmetry? If so, which one(s)?

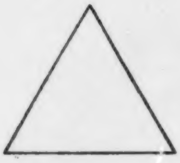
4. Which letters have rotational symmetry. After each letter, tell the amount of the rotation.

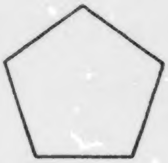
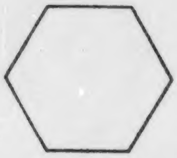
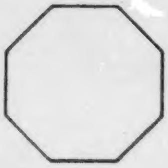
5. Which letters have both rotational and line symmetry?

6. Sketch each figure below. Draw in all lines of symmetry.



7. The figures below are regular polygons, that is, in a given figure all the sides are the same length and all the angles have the same measure. Find the lines of symmetry in each figure.



How is the number of lines of symmetry related to the number of sides in the figure?

8. Which of the regular polygons above has rotational symmetry? Indicate the amount of the rotation, if any.

9. How many lines of symmetry does a circle have?

10. Does a circle have rotational symmetry?

11. Many companies have trademarks that have either line symmetry, rotational

symmetry or both. Many examples can be found in newspapers, magazines and in the yellow pages. Find as many as you can. Cut them out where possible; otherwise sketch or trace the trademarks.

12. Look in an encyclopedia and find as many flags as you can that have line or rotational symmetry. Sketch each flag. Be sure to indicate the country.

Section 2. Plotting Points

In the last chapter you learned that there is a mapping from the set of integers to the points of the number line. The number associated with each point is called the coordinate of that point. Any point can be located if you know its coordinate. But a line has only one dimension and locating points along it soon becomes boring. It would be much more interesting if it were possible to move to positions above and below the number line as well. But how far above and below? Another coordinate is needed; therefore another number line is necessary.

This second number line will cross the first at some point. It seems logical to arrange the lines so that the zero points match. Two possibilities appear below in Fig. 1.

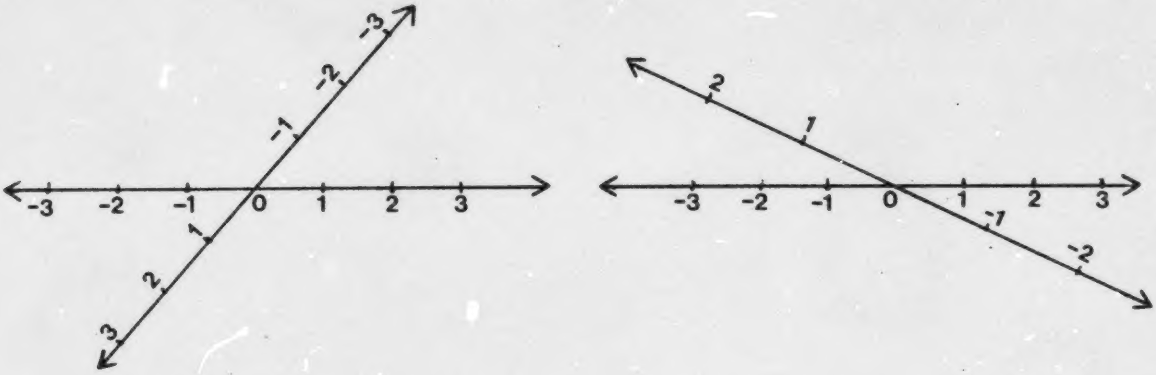


Fig. 1

Several things should be obvious from the diagrams. First, the "new" number line does not have to be at right angles to the first. Second, the distance between 0 and 1 is not necessarily the same on both lines. Third, the part of the new line above the original may be either positive or negative.

If each one of you used a different coordinate system, it would be very confusing. A standard arrangement is essential. First, the two number lines will be placed at right angles to each other, that is they will be perpendicular. Second, the distance between 0 and 1 will be the same on both lines. And third, because most people think of positive as up and negative as down, like a thermometer, the positive direction of the second line will be up and the negative direction down. Fig. 2 shows the "customary" arrangement.

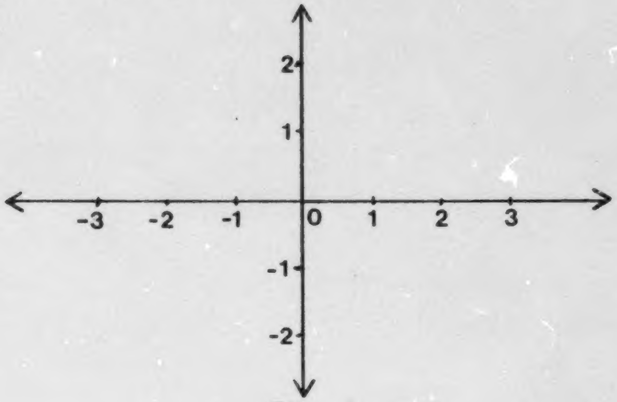


Fig. 2

Special names are given to certain parts of the diagram. The two number lines are called axes. The horizontal line is frequently called the x-axis and the vertical line the y-axis. These two axes separate the plane into four regions, called quadrants. "Quad" means four, just like a quadrilateral has four sides. The quadrants are numbered with Roman numerals in a counterclockwise direction as shown below.

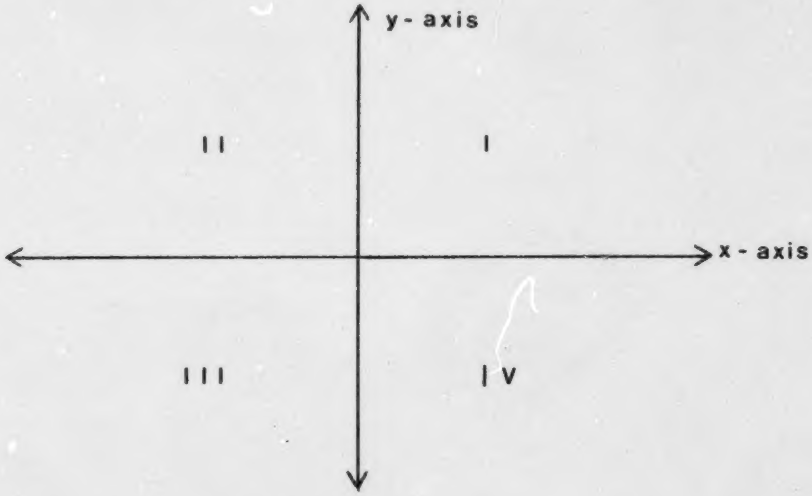


Fig. 3

Each point on the plane will have two numbers associated with it, one for each axis. The first coordinate is the x-coordinate showing the position along the x-axis. The second or y-coordinate shows the position along the y-axis. The pair of coordinates is ordered, so the pair is placed in parentheses, just as before.

One point might be located at (3,5), another at (2, -6) and so forth. In general every point has a location (x,y). The only point that is named specifically is (0,0). It is called the origin. Points are often indicated by capital letters, for example A(3,-2) says that point A is located at (3,-2). To find this location on the graph in Fig. 4, locate 3 on the x-axis, and -2 on the y-axis. Draw lines parallel to the axes through 3 and -2. The point of intersection is A.

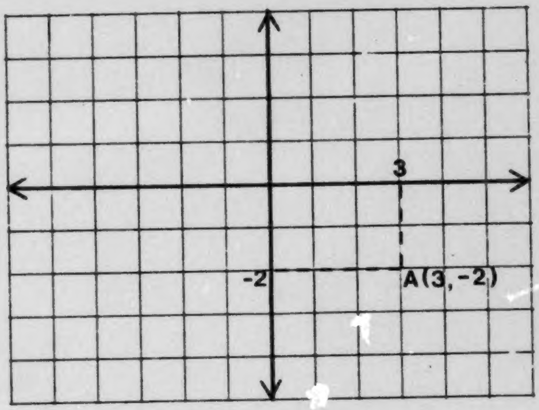
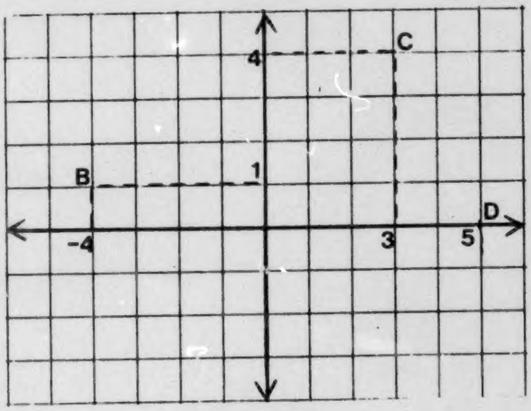


Fig. 4

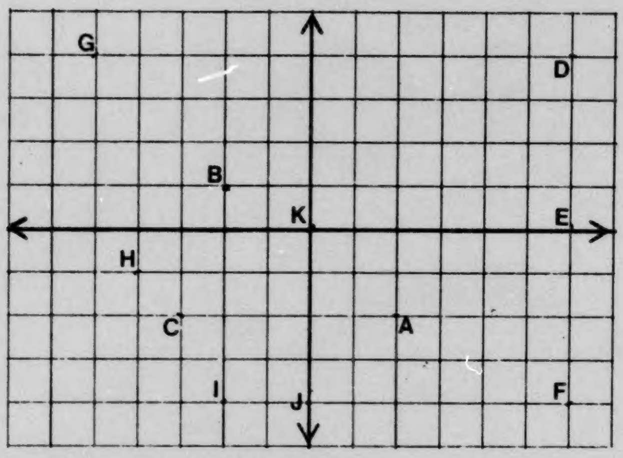
Ex. 1. Locate B(-4,1), C(3,4) and D(5,0).



Because every pair of integers is assigned to a location on the plane, a mapping exists where the domain consists of ordered pairs of integers and the codomain is the set of points on the plane. Notice that the mapping is one-to-one because each point is the image of only one ordered pair. But the mapping is not onto. Many points on the plane are not images of domain elements.

EXERCISES

1. Determine the coordinates of each point on the graph below. List your results in alphabetical order, for example A(2,-2), B(__,__), etc.



2. Locate the points below on the same graph.

A(2,5) B(-3,-1) C(2,-4) D(0,0) E(-3,3)
F(-2,5) G(5,-2) H(-4,0) I(0,5) J(-1,-1)

3. List any three points in the first quadrant.

4. List any three points in the second quadrant.

5. List any three points in the third quadrant.

6. List any three points in the fourth quadrant.

7. Complete this statement: In the first quadrant the first coordinate is a ___________ integer and the second coordinate is a ___________ integer.

8. Write a similar sentence for each of the other three quadrants.

Section 3. Translations

Suppose you are given a graph on which five points have been indicated, as in Fig. 1 below.

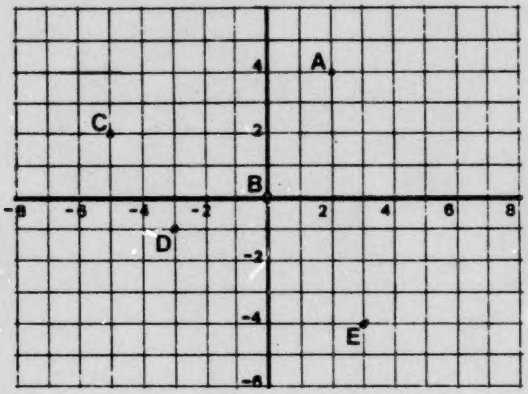


Fig. 1

You are asked to add 3 to the first coordinate of each point. For example, A which has coordinates (2,4) will become A' with coordinates (5,4). The five points are listed below, together with their images. The graph shows the five pairs of points.

A(2,4) → A'(5,4)
B(0,0) → B'(3,0)
C(-5,2) → C'(-2,2)
D(-3,-1) → D'(0,-1)
E(3,-4) → E'(6,-4)

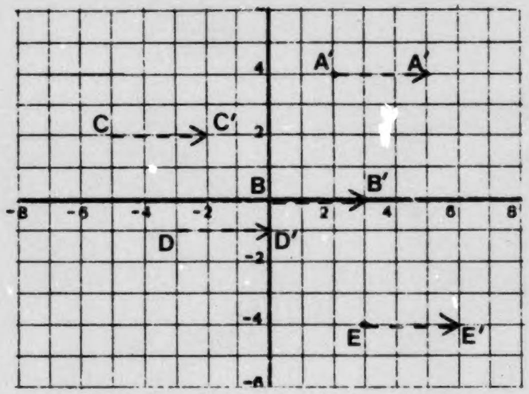


Fig. 2

This listing describes a mapping. The set of points whose coordinates are integers is both the domain and codomain of the mapping. All such points in the plane have images also in the plane. Because every point is the image of only one other point the mapping is both onto and one-to-one.

You know how to describe mappings in many ways, one of which is by writing rules of assignment. For example $n \to n+7$ describes a mapping from W to W, where n represents any element in the set of whole numbers. For the five points listed above, 3 is added to the first element of each ordered pair, while the second element is unchanged. The rule of assignment for this mapping is $(x,y) \to (x+3,y)$, where x and y are any integers.

Study Fig. 2 to see what effect this mapping has on the graph of these points. Each point appears to have slid 3 units to the right along the horizontal. This slide is called a translation. A translation with the rule of assignment $(x,y) \to (x-3,y)$ slides points to the left.

Another translation is described by the rule $(x,y) \to (x,y-4)$. The chart below shows three points A, B, and C and their images A', B' and C' under this translation. On the graph the points are connected to form triangles, ΔABC and $\Delta A'B'C'$.

$A(2,1) \to A'(2,-3)$
$B(6,1) \to B'(6,-3)$
$C(7,3) \to C'(7,-1)$

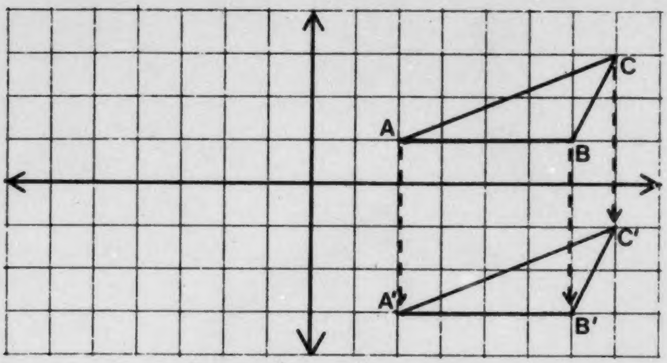


Fig. 3

In this translation the triangle has slid vertically downward. If the rule of assignment had been $(x,y) \to (x,y+4)$, the triangle would have been moved upward. Therefore, adding to the first coordinate slides the points horizontally and adding to the second coordinate slides the points vertically.

What happens if different amounts are added to both coordinates? Consider the translation with rule $(x,y) \to (x-2,y+3)$. Three points and their images are listed below together with the graph.

$(x,y) \to (x-2,y+3)$
$A(2,-2) \to A'(0,1)$
$B(6,-1) \to B'(4,2)$
$C(5,1) \to C'(3,4)$

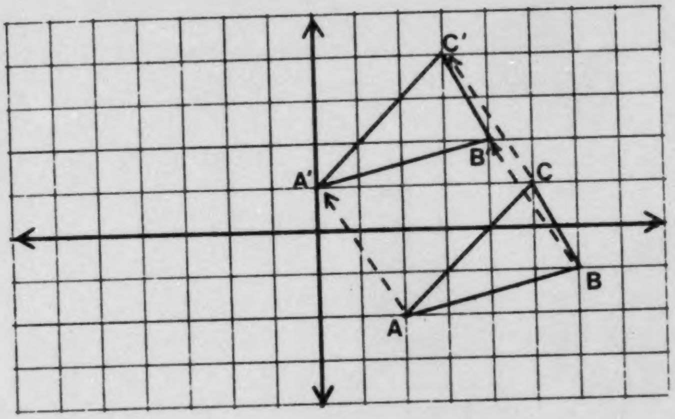


Fig. 4

The image is produced by sliding the triangle to the left and up. The movement to the left results from subtracting 2 from the x-coordinate, the upward movement results from adding 3 to the y-coordinate. The most important thing to realize is that all the points moved precisely the same amount in the same direction. The distance from A to A' is exactly the same as the distance from B to B' and C to C'. If this were not the case, the figure would be distorted. As it is, the image triangle is exactly the same as the original.

A formal definition of translation follows:

> A translation is a mapping of the plane onto itself where
> $(x,y) \to (x+a,y+b)$, where a and b are integers.

A translation is often written $T_{a,b}$, read "translation a,b", where T stands for translation and a and b are the amounts to be added to x and y respectively. Thus $T_{3,-2}$ is equivalent to $(x,y) \to (x+3,y-2)$ and $T_{0,7}$ is equivalent to $(x,y) \to (x,y+7)$.

EXERCISES

1. Graph ΔABC where $A(2,1)$, $B(4,3)$, $C(5,7)$. Locate the image $\Delta A'B'C'$ under translation $(x,y) \to (x+3,y+4)$. List the points and their images.

2. Graph ΔABC where $A(-2,3)$, $B(4,1)$, $C(3,-2)$. Locate the image $\Delta A'B'C'$ under translation $(x,y) \to (x-2,y+1)$. List the points and their images. (The two triangles may overlap.)

3. Graph ΔABC where $A(0,0)$, $B(-2,4)$, $C(1,5)$. Locate the image $\Delta A'B'C'$ under $T_{0,4}$. List the points and their images.

4-7. Graph these points and connect them in order: A(2,-5), B(3,-2), C(-2,1), D(-4,-5). Translate the figure according to each rule below. Put each on a separate graph.

4. $T_{2,5}$ 5. $T_{-3,7}$ 6. $T_{0,0}$ 7. $T_{5,-1}$

Section 4. Translations and Groups

One translation can be performed following another. For example, apply $T_{3,-4}$ to point A(2,1). The image A' has coordinates (5,-3). Now apply $T_{1,3}$ to A'. The image A" has coordinates (6,0). Stated concisely

$$A(2,1) \xrightarrow{T_{3-4}} A'(5,-3) \xrightarrow{T_{1,3}} A''(6,0)$$

The graph in Fig. 1 shows the two translations.

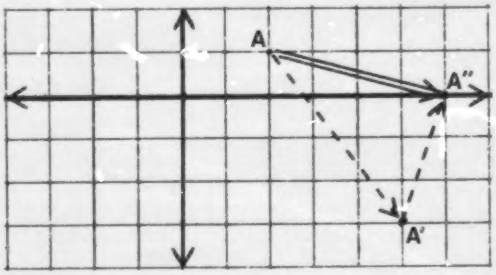


Fig. 1

Would it be possible to write just one translation to move A(2,1) to A"(6,0)? Yes. Locate the double line on the graph. From A to A" the x coordinate increased by 4, and the y coordinate decreased by 1. Therefore

$$(x,y) \longrightarrow (x+4,\ y-1)$$

correctly describes the single translation, $T_{4,-1}$, from A to A". Written symbolically:

$$\begin{tikzcd} A(2,1) \arrow[r, "T_{3,-4}"] \arrow[rr, bend right, dashed, "T_{4,-1}"'] & A'(5,-3) \arrow[r, "T_{1,3}"] & A''(6,0) \end{tikzcd}$$

This combination of translations can be stated very simply using the operation of composition as follows:

$$T_{1,3} \circ T_{3,-4} = T_{3+1,-4+3}$$
$$= T_{4,-1}$$

The single translation equivalent to the two given translations is found by adding the corresponding coordinates. You should recognize the small circle meaning composition. It is the same binary operation you applied to rearranging three books. Now this operation is defined on the set of all translations. Remember that in composition of mappings the order in which the mappings are performed is exactly opposite from most binary operations.

Ex. 1. What single translation is equivalent to $T_{3,-6} \circ T_{-5,2}$?

$$T_{3,-6} \circ T_{-5,2} = T_{-5+3,2+-6}$$
$$= T_{-2,-4}$$

Is $(T_{a,b}, \circ)$ an operational system? Yes, you have just seen that any two translations can be combined into one. In general for all integers a, b, c and d,

$$T_{c,d} \circ T_{a,b} = T_{a+c,b+d}.$$

It is altogether possible that $(T_{a,b}, \circ)$ is a group. Does the system have an identity element? The identity translation must map all points (x,y) to themselves. The translation $T_{0,0}$ adds 0 to both coordinates leaving the points unchanged, as can be seen below.

$$(x,y) \dashrightarrow (x+0,\ y+0)$$
$$(x,y) \dashrightarrow (x,y).$$

In terms of $T_{a,b}$,

$$T_{a,b} \circ T_{0,0} = T_{0,0} \circ T_{a,b} = T_{a,b} \text{ for all translations}$$

Consider the translation $T_{3,-4}$ on a point A(-2,7). The image is A'(1,3) If another translation exists that is the inverse of $T_{3,-4}$, it must map A' to A. The translation $T_{-3,4}$ does just that, that is

$$A(-2,7) \xrightarrow{T_{3,-4}} A'(1,3) \xrightarrow{T_{-3,4}} A(-2,7)$$

or

$$T_{-3,4} \circ T_{3,-4} = T_{0,0}.$$

In general

$$T_{-a,-b} \circ T_{a,b} = T_{a+-a,b+-b} = T_{0,0}$$

Since every integer has an additive inverse, every translation has an inverse. Therefore the operational system has both an identity element and the inverse property. If the system is associative, then it is a group. Consider this example:

$$(T_{2,5} \circ T_{-3,-2}) \circ T_{0,6} = T_{2,5} \circ (T_{-3,-2} \circ T_{0,6})$$

$$T_{-1,3} \circ T_{0,6} = T_{2,5} \circ T_{-3,4}$$

$$T_{-1,9} = T_{-1,9}$$

After checking some other examples in the exercises you should be convinced that $(T_{a,b},\circ)$ is associative and therefore a group.

If the group is commutative, this statement must be true.

$$T_{c,d} \circ T_{a,b} = T_{a,b} \circ T_{c,b}$$

Performing the compositions yields

$$T_{a+c,b+d} = T_{c+a,d+b}$$

Since a,b,c and d are integers and (Z,+) is a commutative group, a+c = c+a and b+d = d+b. Therefore the two parts of the statement above are equivalent and $(T_{a,b},\circ)$ is a commutative group.

EXERCISES

Write the single translation associated with each expression

1. $T_{3,2} \circ T_{4,-6}$

2. $T_{7,-3} \circ T_{-2,-5}$

3. $T_{5,6} \circ T_{2,-8}$

4. $T_{-3,6} \circ T_{2,-5}$

5. Graph triangle ΔABC where $A(2,-5)$ $B(-3,4)$ $C(0,0)$. Find the image $\Delta A'B'C'$ under $T_{4,-2}$ then find the image of $\Delta A'B'C'$ under $T_{2,4}$. Label it $\Delta A''B''C''$. What single translation maps ΔABC to $\Delta A''B''C''$?

6. Graph ΔDEF where $D(2,0)$ $E(0,3)$ $F(2,3)$. Find the image $\Delta D'E'F'$ under $T_{-1,-3}$, then find the image of $\Delta D'E'F'$ under $T_{1,3}$. Label it $\Delta D''E''F''$. What single translation maps ΔDEF to $\Delta D''E''F''$? What is the relationship between $T_{-1,-3}$ and $T_{1,3}$?

Simplify these examples of the associative property.

7. $(T_{3,-2} \circ T_{4,1}) \circ T_{-2,5} = T_{3,-2} \circ (T_{4,1} \circ T_{-2,5})$

8. $(T_{-5,-6} \circ T_{2,4}) \circ T_{3,1} = T_{-5,-6} \circ (T_{2,4} \circ T_{3,1})$

9. $(T_{-3,-4} \circ T_{5,2}) \circ T_{-5,-2} = T_{-3,-4} \circ (T_{5,2} \circ T_{-5,-2})$

10. Write the inverse of each translation.
    a) $T_{3,5}$ b) $T_{4,-3}$ c) $T_{-5,2}$ d) $T_{0,3}$ e) $T_{0,0}$

Section 5. Line Reflections

There are many mappings of the plane onto itself. Collectively they are called plane transformations. You have just studied about one type of transformation called translation. This mapping results from adding given amounts to the coordinates of all points.

The three points and their images listed below are an example of another plane transformation.

A(3,1) ——→ A'(-3,1)
B(1,0) ——→ B'(-1,0)
C(2,-4) ——→ C'(-2,-4)

What relationship exists between the coordinates of each point and its image? The first coordinate is assigned to its additive inverse, that is, $x \longrightarrow -x$, and the second coordinate is left unchanged. Since every integer has an additive inverse and certainly every integer may be mapped to itself, every point on the plane will have an image under this transformation, that is,

$$(x,y) \longrightarrow (-x,y).$$

Study the graph in Fig. 1 to see how the mapping affects the graph of ΔABC.

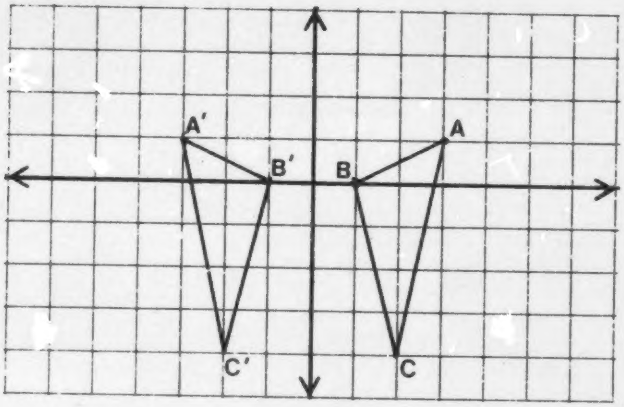


Fig. 1

Certainly the figure has not been translated. In a translation each point slides exactly the same amount. But here the distance from A to A' is much greater than the distance from B to B'. Instead the figure appears to have been flipped over the y-axis. The axis acts like a mirror. ΔA'B'C' is the reflection of ΔABC. The name "line reflection" is given to this type of transformation. This example is called a reflection in the y-axis, written R_y and is defined below

$$(x,y) \longrightarrow (-x,y).$$

Ex. 1. Reflect ΔABC in the y-axis where A(3,1), B(-1,-5), C(2,-4). List the coordinates of $\Delta A'B'C'$. Graph both triangles on a single coordinate system. Draw a line segment joining each point to its image, as shown in Fig. 2.

$(x,y) \longrightarrow (-x,y)$
$A(3,1) \longrightarrow A'(-3,1)$
$B(-1,-5) \longrightarrow B'(1,-5)$
$C(2,-4) \longrightarrow C'(-2,-4)$

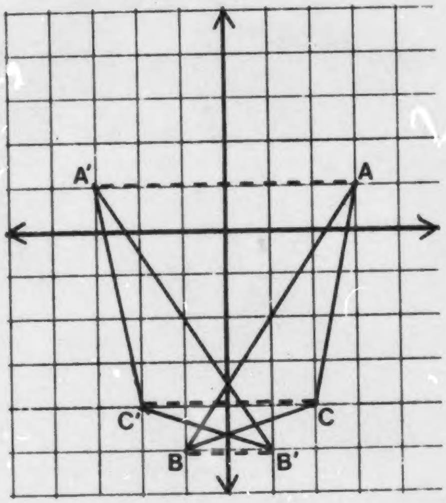


Fig. 2

Find the distance from A to the y-axis; do the same for A'. Both are 3 units from the line of reflection. Points B and B' are each one unit from the axis, and C and C' are each two units from the axis. Any point, not on the line of reflection, and its image are exactly the same distance from the line but on the opposite side of it. Also, the line of reflection is perpendicular to each line segment.

Points on the line of reflection are their own images. To see why this is true, recall the definition for R_y.

$$(x,y) \rightarrow (-x,y)$$

Applying this to a point A(0,-4) on the y-axis yields

$$(0,-4) \rightarrow (-0,-4)$$

or

$$(0,-4) \rightarrow (0,-4)$$

since 0 is its own additive inverse. In a line reflection, all the points on the line are invariant, that is they are their own images under the transformation. This is very different from translations where every point on the plane slides according to the rule $T_{a,b}$.

Other lines on the graph can be designated as lines of reflection. The most obvious is the x-axis. The graph in Fig. 3 shows ΔABC reflected in this axis. Again the image is located an equal distance on the opposite side of the axis.

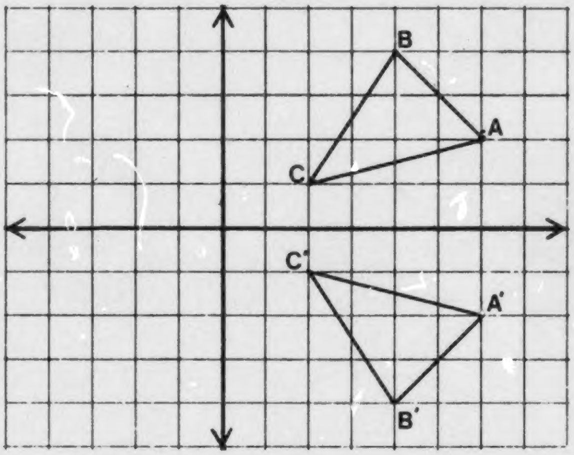


Fig. 3

Listing the points and their images should help you define this type of reflection

$$A(6,2) \rightarrow A'(6,-2)$$
$$B(4,4) \rightarrow B'(4,-4)$$
$$C(2,1) \rightarrow C'(2,-1)$$

In each case the x-coordinate is its own image, while the y-coordinate is mapped to its additive inverse, that is,

$$(x,y) \rightarrow (x,-y)$$

A reflection in the x-axis is written R_x. Notice that R_x assigns the x-coordinate to itself and R_y assigns the y-coordinate to itself. The other coordinates are assigned to their additive inverses.

EXERCISES

1.-6. Find the image of each point under R_x.

1 A(3,-5)   2. B(4,3)   3. C(-3,-2)   4. D(-8,0)   5. E(0,6)   6. F(0,0)

7. List the invariant points.

8.-13. Reflect each point above in the y-axis.

14. List the invariant points.

15. Which point is invariant under both R_x and R_y?

16.-19. Below are the coordinates of four triangles. List the points and their images under R_x. List the points again and the images under R_y. Graph the original triangle and its two images.

16. A(5,3), B(2,3), C(4,4)

17. D(0,1), E(4,2), F(2,-2)

18. G(4,0), H(0,0), I(0,-3)

19. J(1,-1), K(-1,2), L(-3,-3)

Section 6. Other Line Reflections

Another type of line reflection is described by the rule of assignment

$$(x,y) \rightarrow (y,x)$$

This rule is obviously a mapping because the coordinates are just reversed and therefore every point will have an image. Below are several points and their images under this reflection.

$$\begin{aligned} A(3,2) &\rightarrow A'(2,3) \\ B(-1,-4) &\rightarrow B'(-4,-1) \\ C(5,0) &\rightarrow C'(0,5) \\ D(-3,-3) &\rightarrow D'(-3,-3) \end{aligned}$$

Notice that D(-3,-3) maps to itself. This true of every point where the x and y coordinates are the same. For example, (2,2), (-5,-5) and (0,0) are all invariant under this line reflection. Recall that in previous line reflections only the points on the line of reflection were invariant. This should lead you to suspect that this new line of reflection contains points (0,0), (2,2) and (-5,-5) and all points (x,y) where x=y. To test this hypothesis, graph several points where the coordinates are the same, as shown in Fig. 1 below.

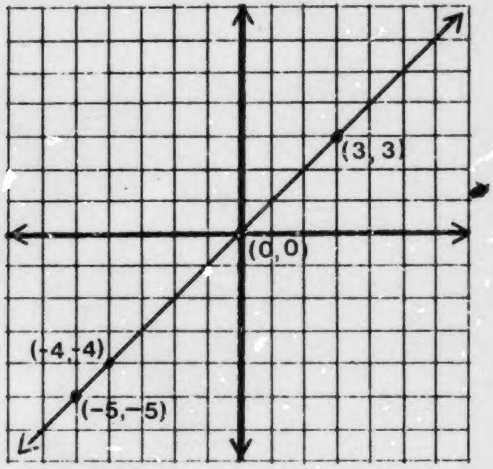


Fig. 1

The diagonal line through the points is called D1 and separates quadrants I and III into two equal regions. Now add to the graph the points A, B, C and D above and their respective images. Join each point to its image.

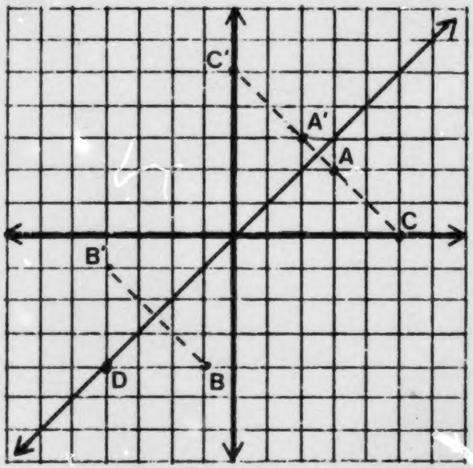


Fig. 2

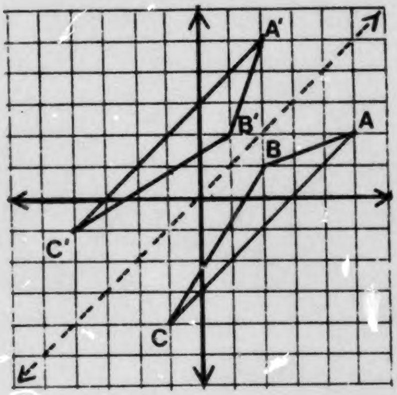
Are each point and its image equal distances on opposite sides of the line of reflection? Yes. Is it also true that each line segment joining a point and its image is perpendicular to the line of reflection? Yes. It is therefore correct to conclude that the rule of assignment

$$(x,y) \rightarrow (y,x)$$

describes a reflection in the diagonal D1. This is written R_{D1}.

Ex. 1 Find the image of ΔABC where $A(5,2)$, $B(2,1)$, $C(-1,-4)$ under R_{D1}.

$A(5,2) \rightarrow A'(2,5)$
$B(2,1) \rightarrow B'(1,2)$
$C(-1,-4) \rightarrow C'(-4,-1)$



There is another diagonal line which separates quadrants II and IV. The graph in Fig. 3 shows the line and several points on it.

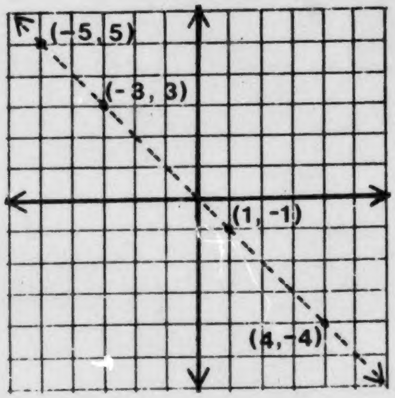


Fig. 3

In every case the coordinates are additive inverses of each other. This line is called D2 and a reflection in the line is denoted R_{D2}. To determine the rule that defines this line reflection, select several points not on the line, for example A(4,-2), B(2,1), C(-3,0), and D(-3,4). Graph these points. To locate their images remember that the image is an equal distance on the opposite side of the line of reflection and that the segment joining the point and its image is perpendicular to the line of reflection. These facts lead to the graph in Fig. 4.

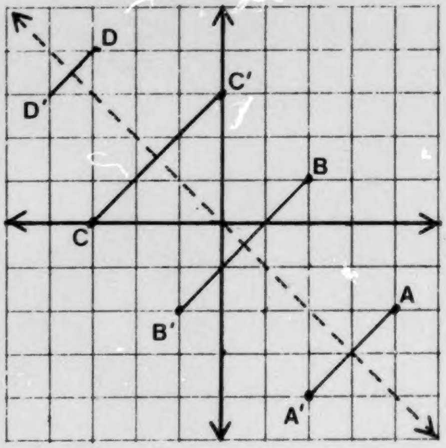


Fig. 4

The list of points and their images, read directly from the graph, should enable you to write the rule for this mapping.

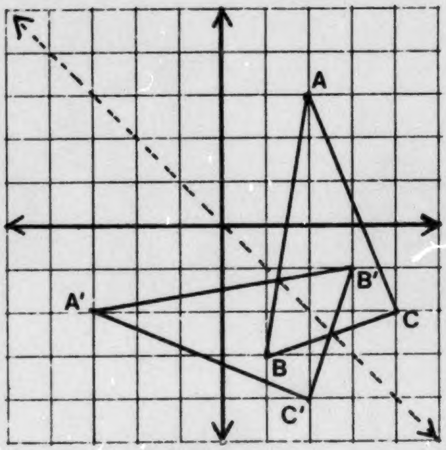
A(4,-2) → A'(2,-4)
B(2,1) → B'(-1,-2)
C(-3,0) → C'(0,3)
D(-3,4) → D'(-4,3)

Notice that, as in R_{D1}, the coordinates are reversed. But here the signs are also changed. In general, the line reflection, R_{D2} maps points according to the rule

$$(x,y) \rightarrow (-y,-x).$$

Ex. 2. Find the image under R_{D2} of ΔABC where A(2,3), B(1,-3), C(4,-2).
Graph both ΔABC and $\Delta A'B'C'$.

A(2,3) → A'(-3,-2)
B(1,-3) → B'(3,-1)
C(4,-2) → C'(2,-4)



Any line in the plane can be used as a line of reflection. The rules governing these mappings are rather complex and require a greater understanding of algebra. The four line reflections you have learned are the simplest and most important. They are summarized below:

Reflection in the x-axis,	R_x:	(x,y) → (x,-y)
Reflection in the y-axis,	R_y:	(x.y) → (-x,y)
Reflection in first diagonal,	R_{D1}:	(x,y) → (y,x)
Reflection in second diagonal,	R_{D2}:	(x,y) → (-y,-x)

Two translations can be combined into one translation by the operation of composition. Can two line reflections be combined into one line reflection? You have seen that a line reflection flips a triangle over. A second flip would make the triangle right side up again. Since no single line reflection leaves a triangle right side up, two flips cannot be combined into one, that is, two line reflections are not equivalent to a single line reflection. For this reason, the set of line reflections and the operation of composition is not an operational system.

EXERCISES

Find the image of each point below under R_{D1}.
1. A(-3,5)   2. B(4,3)   3. C(5,-2)   4. D(-4,-3)

5. Make a graph showing each point in 1-4 and its image. Join each point to its image by a line segment.

Find the image of each point below under R_{D2}.

6. R(3,-4)　　7. S(-2,-1)　　8. T(-3,0)　　9. U(2,5)

10. Make a graph showing each point in 6-9 and its image. Join each point to its image.

11. Given the point A(-3,2), find its image under each of the following line reflections

a) R_x　　b) R_y　　c) R_{D1}　　d) R_{D2}

12. Graph A and its four images. Label each point carefully.

13. Given ΔBCD where B(3,-5), C(5,-1), D(1,2), find the image ΔB'C'D' under R_{D1}. Graph both triangles.

14. Given ΔRST where R(0,3), S(2,-2), T(5,0), find the image ΔR'S'T' under R_{D2}. Graph both triangles.

Section 7. Rotations

Think of a ball on the end of a string. Twirl it. The ball rotates in a circular path around the fixed end of the string. The ball will always be the same distance from the center determined by the length of the string as seen in Fig. 1.

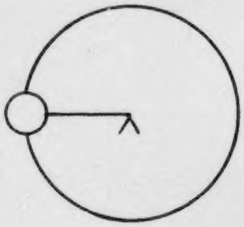


Fig. 1

Rotations in the plane are very similar. The origin is the center of the rotation, and points revolve around it at a constant distance from it. The point A in Fig. 2 is two units from the origin. It may be mapped to any other point on the circle of radius 2. Point B is three units from the origin and is assigned to any point on its circle.

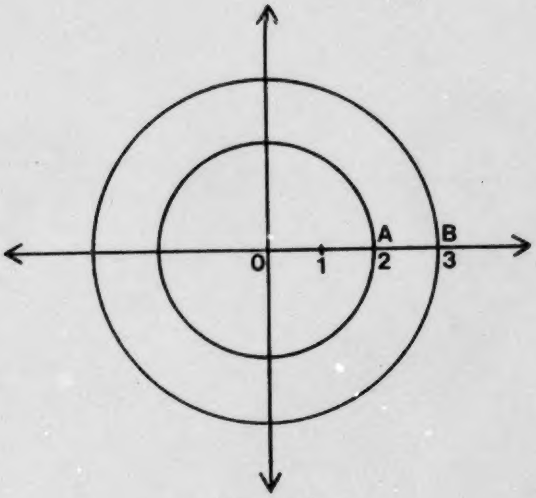


Fig. 2

When the plane rotates all points on the plane, move a given amount in a counter-clockwise direction. Rotations are measured in degrees, and can have any value between 0 and 360. Only three rotations will be studied here in detail: rotations of 90, 180 and 270 degrees.

Begin by rotating the two points A(2,0) and B(0,3) by 90 degrees. The image of A, still 2 units from the origin, will be on the y-axis, A'(0,2); the image of B will be along the x-axis at (-3,0), as shown in Fig. 3 below.

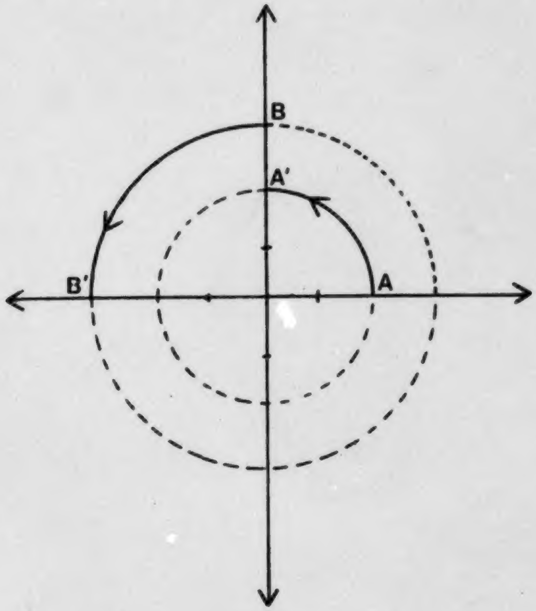
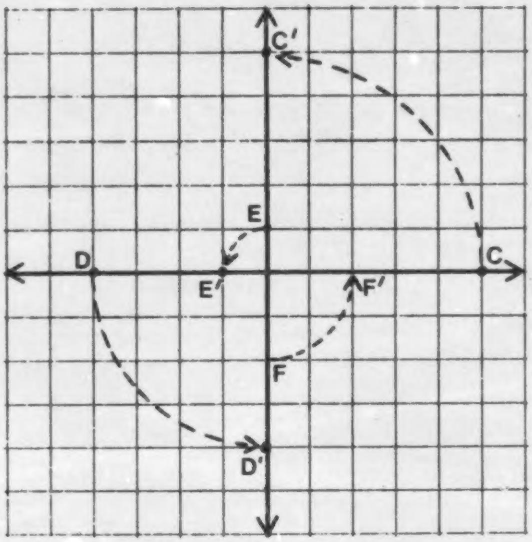


Fig. 3

Ex. 1. Graph these points, C(5,0), D(-4,0), E(0,1), F(0,-2) and their images under a rotation of 90 degrees. The coordinates of the image points, read from the graph are listed below:

C(5,0) → C'(0,5)
D(-4,0) → D'(0,-4)
E(0,1) → E'(-1,0)
F(0,-2) → F'(2,0)



From this list you probably have some idea of the rule of assignment for this mapping. Certainly the two coordinates are reversed, but there is also a sign change in some cases. Look carefully at points C and D. No sign change occurred in the original x-coordinates when the order of the coordinates was reversed.

C(5,0) → C'(0,5)

D(-4,0) → D'(0,-4)

Now study points E and F. Each original y-coordinate is assigned to its additive inverse in both cases.

E(0,1) → E'(-1,0)

F(0,-2) → F'(2,0)

Therefore a rotation of 90 degrees has the following rule of assignment.

(x,y) → (-y,x)

This rule can be applied to all points on the plane, not just those along the axes. The symbol P_{90} is used to denote this rotation. R is not used to avoid confusion with line reflections; P refers to the point (0,0) which is invariant in any rotation.

Ex. 2 Find the image of each point listed below using the rule for rotations of 90 degrees. Graph the points and their images.

A(5,-2) → A'(2,5)
B(-3,1) → B'(-1,-3)
C(-4,-2) → C'(2,-4)
D(2,2) → D'(-2,2)

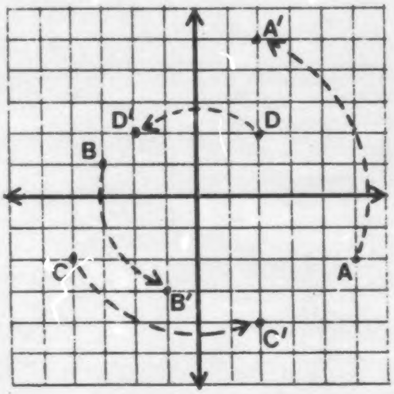


Figure 4 shows A(5,0) and its image A'(-5,0) under a rotation of 180 degrees.

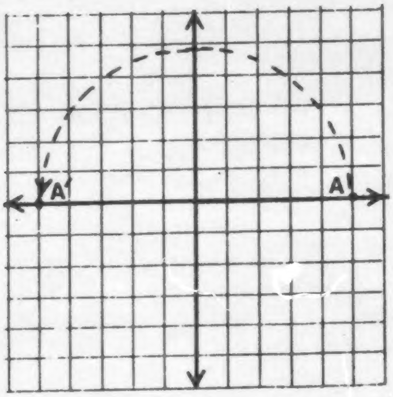


Fig. 4

Since 180 degrees is exactly half the circle, the line segment from A to A' is a diameter of the circle. A' is located on the opposite side of the origin from A. This fact makes it quite simple to locate images of points rotated 180 degrees. The graph in Fig. 5 shows several points and their images. The coordinates, read directly from the graph are listed below.

B(0,4) → B'(0,-4)
C(2,2) → C'(-2,-2)
D(5,-1) → D'(-5,1)

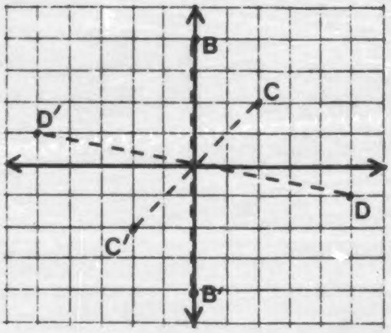


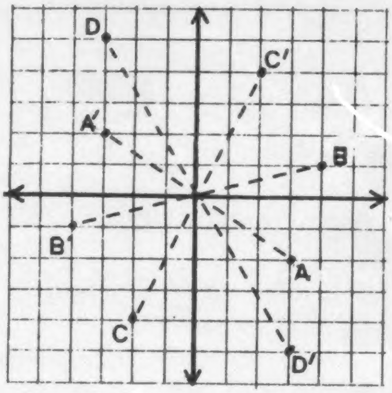
Fig. 5

In this mapping, the coordinates are not reversed, but they are both assigned to their additive inverses. This leads to the following rule of assignment for rotations of 180 degrees, denoted P_{180}.

$$(x,y) \rightarrow (-x,-y).$$

Ex. 3 Using the plane transformation P_{180}, find the images of the points listed below. Graph both the points and their images.

A(3,-2) → A'(-3,2)
B(4,1) → B'(-4,-1)
C(-2,-4) → C'(2,4)
D(-3,5) → D'(3,-5)



Another way to perform a rotation of 180 degrees is to rotate 90 degrees and then rotate 90 degrees again. The binary operation composition is used for rotations in much the same way as for translations. The double rotation is written as follows

$$P_{90} \circ P_{90} = P_{180}$$

Test the accuracy of this statement by selecting a point, for example, A(3,2) and finding its image after two 90 degree rotations. Recall the rule for rotations of 90 degrees: $(x,y) \rightarrow (-y,x)$. The first 90 degree rotation maps A to A' as follows

$$A(3,2) \rightarrow A'(-2,3).$$

The second rotation maps A' to A".

$$A'(-2,3) \rightarrow A''(-3,-2).$$

Combining the two rotations yields

$$A(3,2) \rightarrow A'(-2,3) \rightarrow A''(-3,-2)$$

Compare A and A".

$$A(3,2) \rightarrow A''(-3,-2).$$

This exactly fits the rule for rotations of 180 degrees, $(x,y) \rightarrow (-x,-y)$.

This and other examples in the exercises should lead you to accept the truth of the statement

$$P_{90} \circ P_{90} = P_{180}$$

This ability to combine rotations can be used to determine the rule of assignment for rotations of 270 degrees as follows

$$P_{180} \circ P_{90} = P_{270}$$

Select a point, for example B(-1,4) and find its image under $P_{180} \circ P_{90}$. The rotation of 90 degrees which is performed first, maps B to B'.

$$B(-1,4) \rightarrow B'(-4,-1)$$

The image of B' under P_{180} is

$$B'(-4,-1) \rightarrow B''(4,1)$$

Combining these two mappings yields

$$B(-1,4) \rightarrow B'(-4,-1) \rightarrow B''(4,1)$$

The graph in Fig. 6 shows B, B' and B". The arrows indicate the amount of rotation.

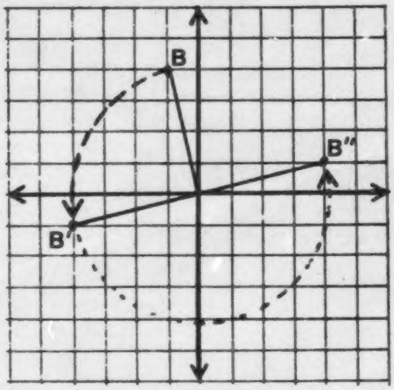


Fig. 6

You should be able to see from the graph that the total rotation is 270 degrees. To write a rule of assignment for the mapping, compare B and B".

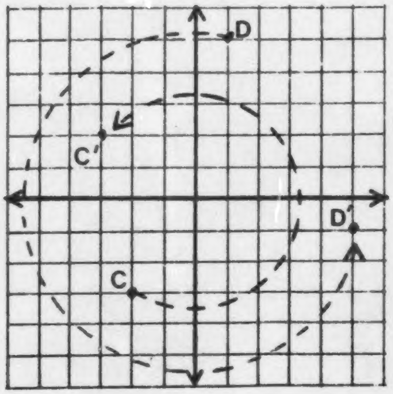
B(-1,4) → B"(4,1)

Notice that the coordinates are reversed and the x-coordinate is assigned to its additive inverse. This observation leads to the following rule of assignment for P_{270}

(x,y) → (y,-x)

Ex. 4 Find the image of C(-2,-3) and D(1,5) under P_{270}. Graph the points and their images.

C(-2,-3) → C'(-3,2)
D(1,5) → D'(5,-1)



Using the binary operation of composition any two rotations can be combined by adding the number of degrees in each rotation. You have seen two examples of this already.

$$P_{90} \circ P_{90} = P_{180}$$

$$P_{180} \circ P_{90} = P_{270}$$

The rotation of 270 degrees is also equivalent to

$$P_{90} \circ P_{180} = P_{270}$$

You will have the opportunity to check this in the exercises. If two rotations are combined, whose sum is 360 degrees, the final image is the same as the original point. A rotation of 360 degrees is the same as a rotation of 0 degrees, or no rotation at all; symbolically,

$$P_{360} = P_0$$

For combined rotations greater than 360 degrees, simplify the sum just as you did in finite systems. In fact the set of rotations is finite and can be thought of as Z_{360}.

Ex. 5 Simplify $P_{180} \circ P_{270}$

$$\begin{aligned} P_{180} \circ P_{270} &= P_{270+180} \\ &= P_{450} \text{ (subtract 360)} \\ &= P_{90} \end{aligned}$$

Any two rotations can be combined to yield another rotation as follows

$$P_b \circ P_a = P_{a+b}.$$

Therefore an operational system exists. It is called $(P_a, \circ)$ where a is any number of degrees in Z_{360}. Whether or not this system forms a group is determined in the exercises that follow.

EXERCISES

List the rules of assignment for each rotation

1. P_{90} 2. P_{180} 3. P_{270} 4. P_0

Find the image of A(2,-5) under each rotation

5. P_{90} 6. P_{180} 7. P_{270} 8. P_0

Find the image of B(-3,-4) under each rotation

9. P_{90} 10. P_{180} 11. P_{270} 12. P_0

13. Graph B and its four images on one graph.

Find the image of each point under P_{270}.

14. (6,-5) 15. (-3,-3) 16. (5,0) 17. (0,-3)

Simplify the following expressions.

18. $P_{270} \circ P_{270}$ 19. $P_{180} \circ P_{180}$ 20. $P_{270} \circ P_{90}$

21. $P_{90} \circ P_{270}$ 22. $P_{180} \circ P_{90}$ 23. $P_{90} \circ P_{180}$

24. $P_0 \circ P_{90}$ 25. $P_{90} \circ P_0$ 26. $P_0 \circ P_0$

27. Problems 22 and 23, among others, illustrate a very important property you have studied. Name it. Recall when you first encountered composition in rearranging books. Did that system have this property?

28. Make a table for the set of rotations $\{P_0, P_{90}, P_{180}, P_{270}\}$.

$\circ$	P_0	P_{90}	P_{180}	P_{270}
P_0				
P_{90}				
P_{180}				
P_{270}				

29. Is there an identity element? If so, name it.

30. Does the system have the inverse property? If so, name the inverse of each element.

31. Is the system associative? Give two examples.

32. Does this set of rotations form a group for the operation of composition?

33. Is the group commutative? How do you tell from the table?

Section 8. A Transformational Group

In all, you have learned three major types of plane transformations, line reflections, rotations and translations. Setting the last aside for a moment, the four line reflections and 3 rotations are listed below together with their rules of assignment.

R_x: $(x,y) \to (x,-y)$
R_y: $(x,y) \to (-x,y)$
R_{D1}: $(x,y) \to (y,x)$
R_{D2}: $(x,y) \to (-y,-x)$
P_{90}: $(x,y) \to (-y,x)$
P_{180}: $(x,y) \to (-x,-y)$
P_{270}: $(x,y) \to (y,-x)$

By selecting a single point, for example A(4,1), and finding its image under each transformation, an interesting pattern emerges, as shown on the graph in Fig. 1.

R_x: $(4,1) \to (4,-1)$
R_y: $(4,1) \to (-4,1)$
R_{D1}: $(4,1) \to (1,4)$
R_{D2}: $(4,1) \to (-1,-4)$
P_{90}: $(4,1) \to (-1,4)$
P_{180}: $(4,1) \to (-4,-1)$
P_{270}: $(4,1) \to (1,-4)$

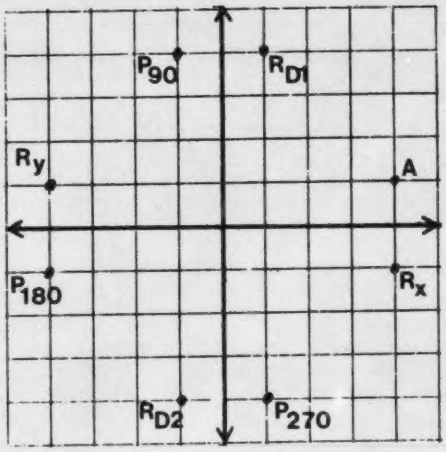


Fig. 1

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Notice the symmetric arrangement of the images. In the plane there are only these eight points which have as coordinates some combination of 1 and 4, or their additive inverses. All the points also lie on one circle with center at the origin, as shown in Fig. 2.

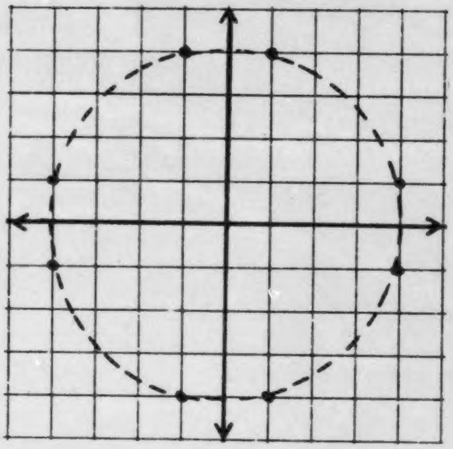


Fig. 2

It is possible that these seven transformations, together with the rotation of 0 degrees as the identity transformation, form an operational system. The obvious binary operation is composition. Earlier two line reflections were found not to be equivalent to a single line reflection. Therefore two line reflections must be equivalent to some rotation. Otherwise the set is not possibly an operational system. Begin with $R_y \circ R_x$, and the point A(4,1).

A reflection in the x-axis maps A to A' as follows

$$A(4,1) \rightarrow A'(4,-1)$$

A reflection in the y-axis moves A' to A"

$$A'(4,-1) \rightarrow A''(-4,-1)$$

Combining the two line reflections yields

$$A(4,1) \rightarrow A'(4,-1) \rightarrow A''(-4,-1)$$

Compare A and A"

$$A(4,1) \rightarrow A''(-4,-1)$$

This is equivalent to a rotation of 180 degrees, therefore

$$R_y \circ R_x = P_{180}$$

Because this pair of reflections is equivalent to a rotation, all other combinations must be checked.

Ex. 1 Find the transformation equivalent to $R_{D1} \circ R_y$.

Select the point B(-3,1) at random.

R_y: B(-3,1) → B'(3,1)

R_{D1}: B'(3,1) → B"(1,3)

$R_{D1} \circ R_y$: B(-3,1) → B"(1,3)

The final mapping is equivalent to a rotation of 270 degrees; therefore

$$R_{D1} \circ R_y = P_{270}.$$

Ex. 2. $P_{90} \circ R_x = ?$ Select C(3,2) at random.

R_x: C(3,2) → C'(3,-2)

P_{90}: C'(3,-2) → C"(2,3)

$P_{90} \circ R_x$: C(3,2) → C"(2,3)

The final reflection is equivalent to R_{D1}; therefore

$$P_{90} \circ R_x = R_{D1}.$$

Continue to combine various pairs of reflections and rotations. Organize your results in the table shown below.

EXERCISES

1. Complete the table

$\circ$	P_0	P_{90}	P_{180}	P_{270}	R_x	R_y	R_{D1}	R_{D2}
P_0								
P_{90}								
P_{180}								
P_{270}								
R_x								
R_y								
R_{D1}								
E_{D2}								

2. Does the table illustrate an operational system?

3. Is there an identity element? Name it.

4. Does the system have the inverse property?
List the inverse of each element if one exists.

5. Is the system associative? Give at least two examples.

6. Is the system a group?

7. Is the system commutative?

Section 9. Isometries

All of the mappings discussed so far have one important property in common. The image and the original figure are identical in size and shape. If you cut out the original figure, it will fit exactly on top of its image. These types of mappings are called isometries. The reason that there is no distortion is because the distance between pairs of points does not change. If two points are, say, three units apart, they will remain three units apart under a translation, a line reflection, or a rotation.

In the last section you discovered a non-commutative group using line reflections and rotations, both of which are isometries. Some familiar geometric shapes can be mapped onto themselves by certain line reflections and rotations to form other operational systems and possible groups.

One such figure is a rectangle. As each isometry is performed, the rectangle will move, but its final position will be exactly as it was in the beginning. For example, if the rectangle was in the position shown in Fig. 1 initially, each isometry must return it to this position. Neither position in Fig. 2 is acceptable.

Fig. 1

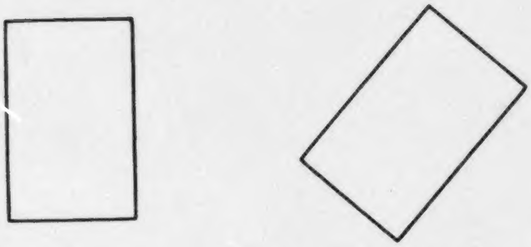


Fig. 2

A rectangle has two lines of symmetry as you learned earlier. They are also the lines of reflection for the rectangle, and are labeled ℓ_1 and ℓ_2 as shown in Fig. 3.

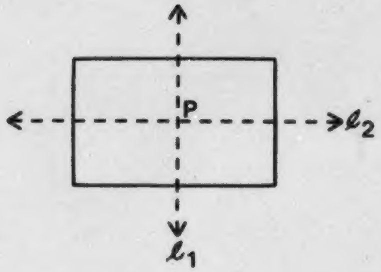


Fig. 3

Cut out a rectangle similar to the one shown here. Hold the midpoints of two opposite sides. Flip the rectangle. Is it in the same position as before? If not, repeat the process. When the flip is complete you should not be able to tell that a transformation has occurred. Now reflect the rectangle in the other line. Again the figure should appear unchanged.

Another motion that leaves the rectangle's position unchanged is a rotation of of 180 degrees around P, the point of intersection of ℓ_1 and ℓ_2. The last isometry to be performed on the rectangle is the identity transformation, e, which is equivalent to a rotation of zero degrees.

Now label your rectangle as shown in Fig. 4.

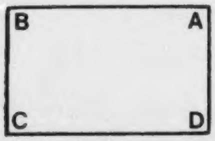


Fig. 4

Turn it over and label it so that the two A's are at the same vertex, the two B's, the two C's and the two D's. Turn your figure over again. This position is the starting position. Each of the four isometries of the rectangle is shown in Fig. 5. Practice each one several times. Be sure to return to the starting position after each transformation.

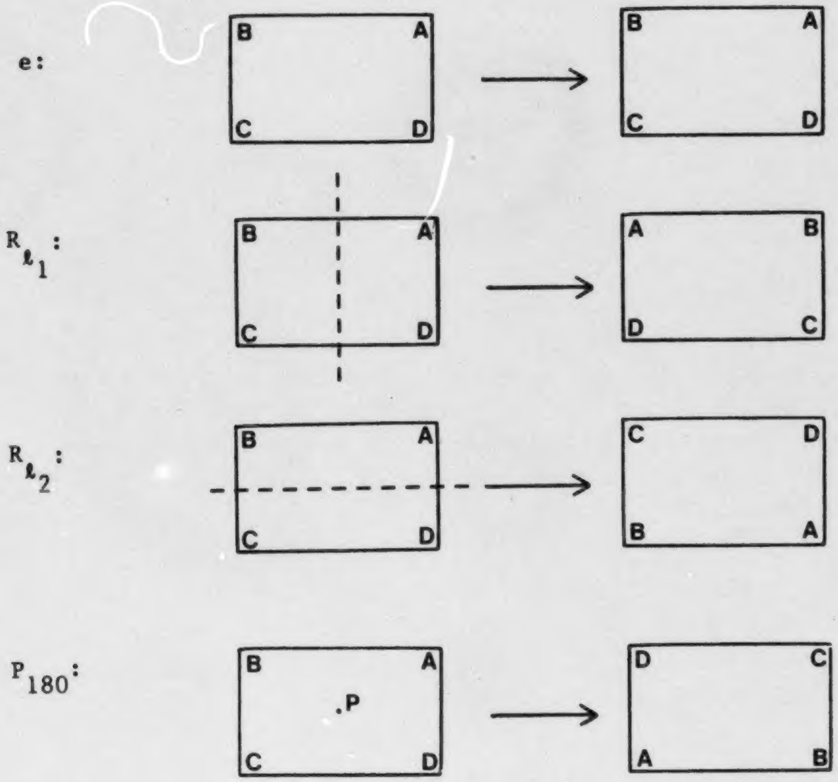
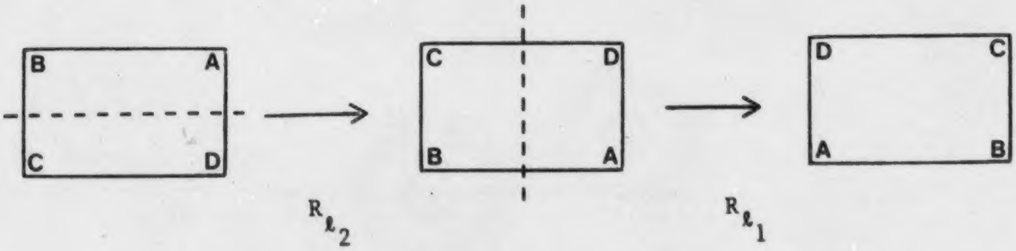


Fig. 5

The binary operation composition is again used to combine two isometries into one. Study these examples very carefully. Use the results to help you to complete the table in the exercises.

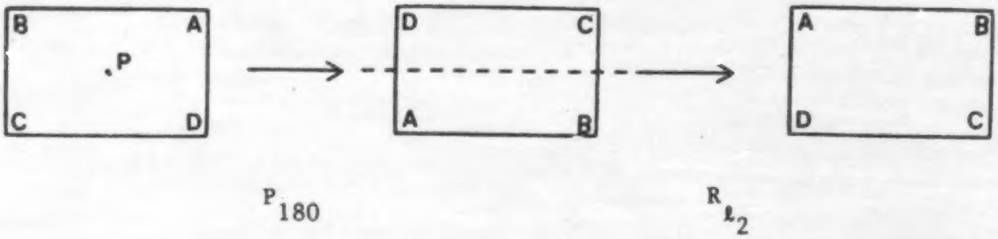
Ex. 1. $R_{\ell_1} \circ R_{\ell_2}$



The final result is the same as a single rotation of 180 degrees; therefore,

$$R_{\ell_1} \circ R_{\ell_2} = P_{180}$$

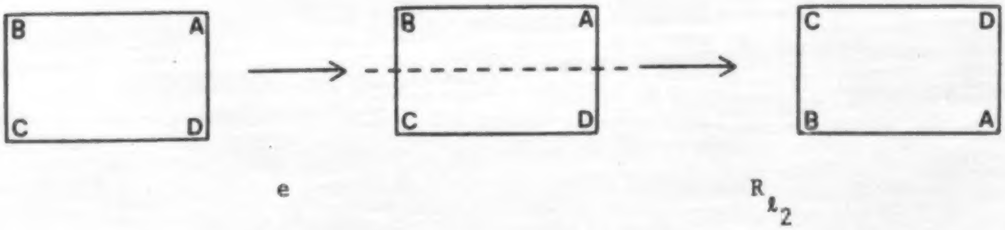
Ex. 2. $R_{\ell_2} \circ P_{180}$



The final result is equivalent to a R_{ℓ_1}; therefore,

$$R_{\ell_2} \circ P_{180} = R_{\ell_1}$$

Ex. 3. $R_{\ell_2} \circ e$



$$e \circ R_{\ell_2} = R_{\ell_2}$$

The rest of the table can be completed in a similar manner.

Another geometric figure of interest is the equilateral triangle. The set of isometries of the triangle must again leave the triangle unchanged so that if it were not for the labeling of the vertices, it would be impossible to determine if some transformation had occurred or not.

First consider the possible rotations that leave ΔABC unchanged. Obviously a rotation of zero degrees will not alter the figure. As before this rotation is the identity element and will be denoted by the letter e. A rotation that moves the triangle 1/3 of a full turn in a counter-clockwise direction is also an isometry. It is illustrated in Fig. 6 below.

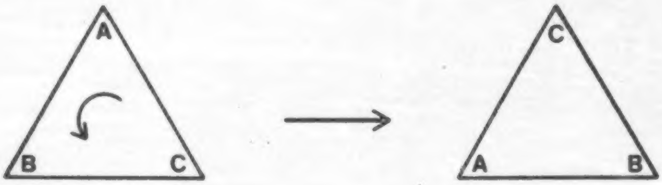


Fig. 6

This rotation will be called P_{120} because 120 degrees is one-third of the whole circle or 360 degrees. A rotation of 240 degrees, P_{240}, is also an isometry as shown below.

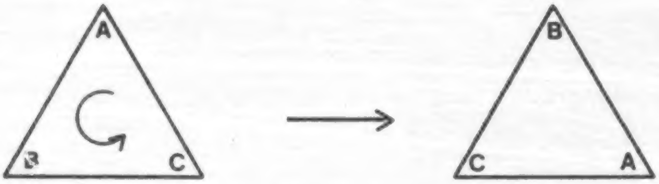


Fig. 7

This mapping rotates the triangle two thirds of a full turn.

In addition to the rotations an equilateral triangle has three lines of symmetry. These lines of reflection are isometries for the triangle. Figure 8 shows these lines.

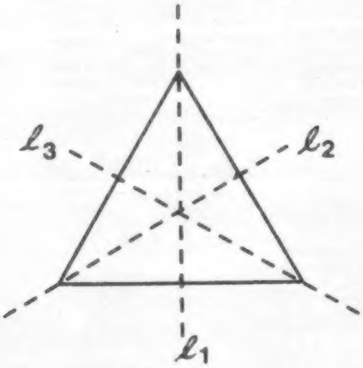


Fig. 8

The lines are always in the same position regardless of what letters are at the vertices. For example, ℓ_1 is always vertical. It does not necessarily pass throught point A. The lines of reflection are written R_{ℓ_1}, R_{ℓ_2}, R_{ℓ_3}.

When combined with the rotations the following set of six elements is formed $\{e, P_{120}, P_{240}, R_{\ell_1}, R_{\ell_2}, R_{\ell_3}\}$. In the exercises you will be asked to construct a table for this set. Before you begin cut out an equilateral triangle. Label it ABC as shown.

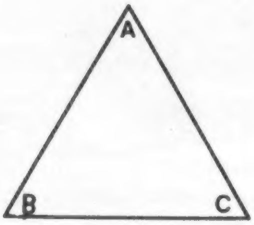


Fig. 9

Turn the triangle over. Label the back in the same way you labeled the back of the rectangle. Figure 10 below shows the six possible isometries of the triangle. Practice each one several times.

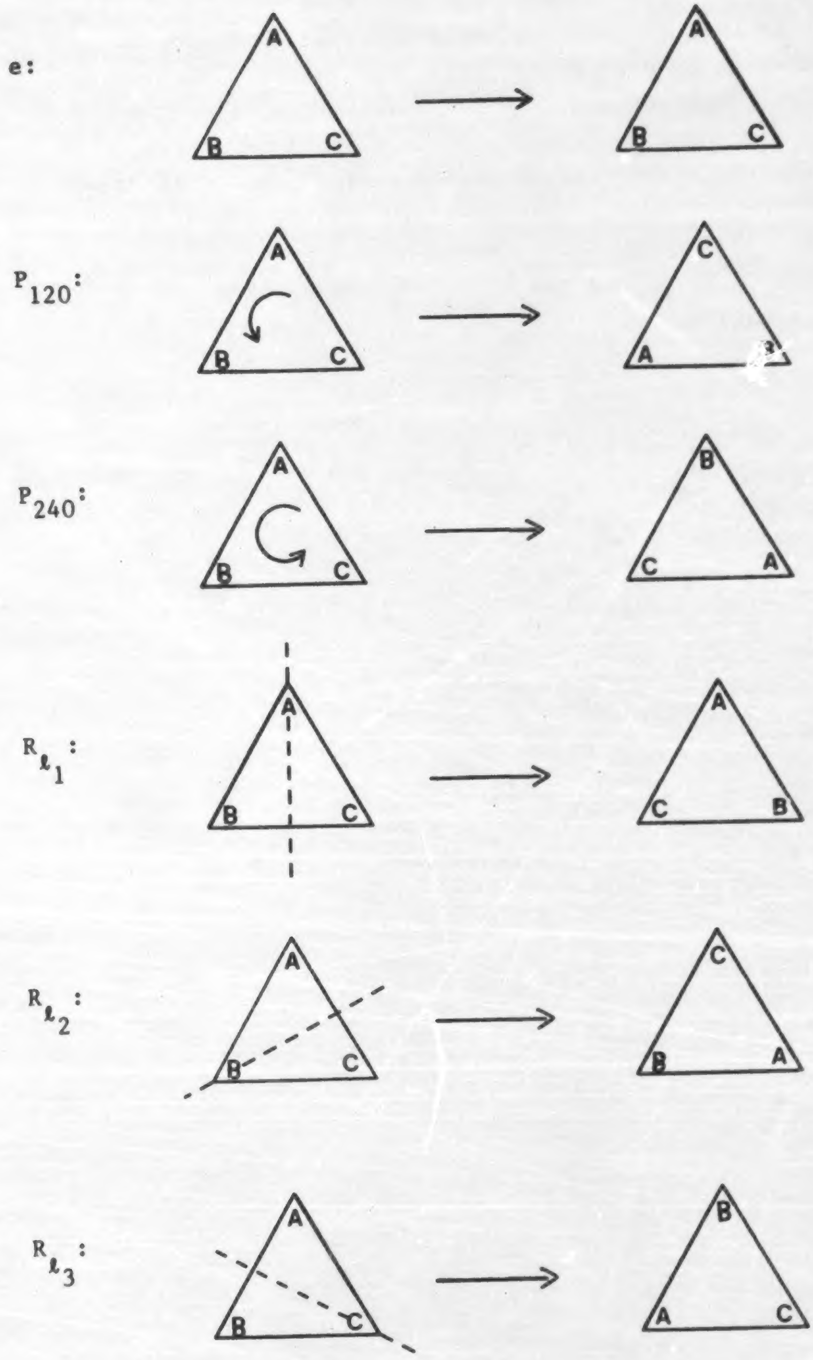
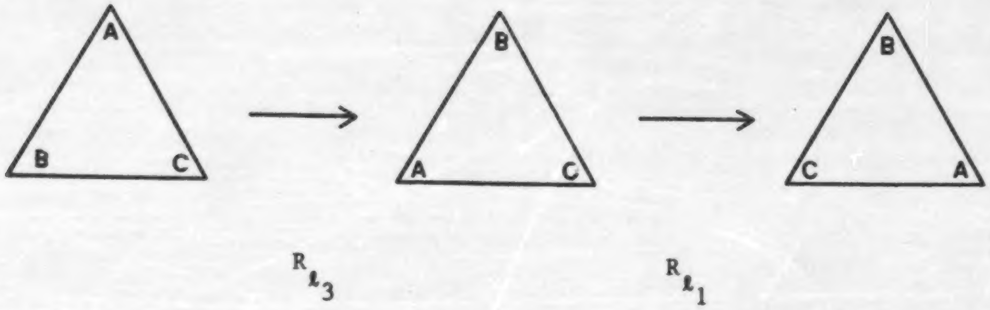

e:
A
B
C
A
B
C
P_{120}:
A
B
C
C
A
B
P_{240}:
A
B
C
B
C
A
R_{ℓ_1}:
A
B
C
A
C
B
R_{ℓ_2}:
A
B
C
C
B
A
R_{ℓ_3}:
A
B
C
B
A
C

Fig. 10

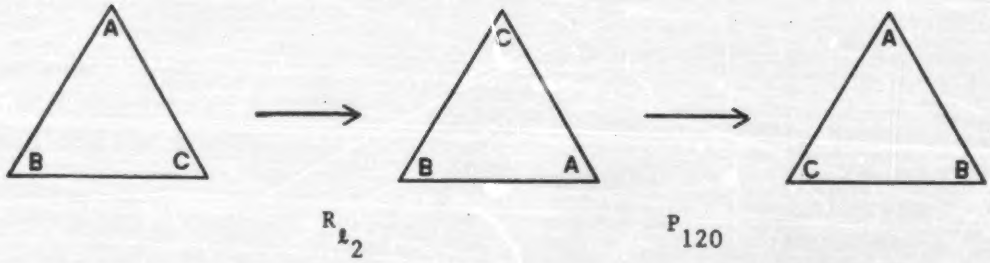
Now study these examples very carefully. Use your triangle to perform the different isometries.

Ex. 1 $R_{\ell_1} \circ R_{\ell_3}$



Therefore $R_{\ell_1} \circ R_{\ell_3} = P_{240}$.

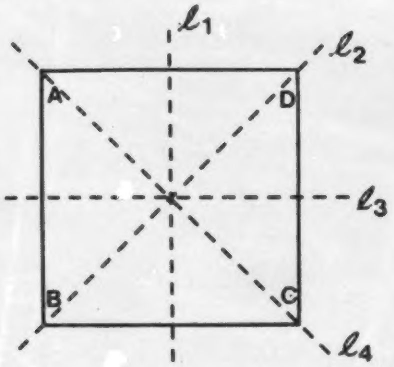
Ex. 2. $P_{120} \circ R_{\ell_2}$



Hence $R_{\ell_2} \circ P_{120} = R_{\ell_1}$.

Use this method to help you complete the table.

A square can be mapped onto itself by eight isometries: the identity e; the three rotations P_{90}, P_{180} and P_{270}; and the four line reflections R_{ℓ_1}, R_{ℓ_2}, R_{ℓ_3}, R_{ℓ_4}. The figure below shows the position of the lines of reflection and the labeling of the vertices in the initial position. In the exercises you will complete a table showing the results of combining these isometries.



EXERCISES

1. Complete the table for the isometries of a rectangle.

$\circ$	e	R_{ℓ_1}	R_{ℓ_2}	P_{180}
e				
R_{ℓ_1}				
R_{ℓ_2}				
P_{180}				

2. Does the table illustrate an operational system?

3. Does the system have an identity element? Name it.

4. Does the system have the inverse property? Name the inverse of each element if one exists.

5. Is the system commutative? Why?

6. Is the system associative? Check these two examples.

a) $(R_{\ell_1} \circ R_{\ell_2}) \circ P_{180} = R_{\ell_1} \circ (R_{\ell_2} \circ P_{180})$

b) $(R_{\ell_2} \circ P_{180}) \circ R_{\ell_1} = R_{\ell_2} \circ (P_{180} \circ R_{\ell_1})$

7. Is the system a group? Is it a commutative group?

8. Complete the table for the isometries of an equilateral triangle.

$\circ$	e	P_{120}	P_{240}	R_{ℓ_1}	R_{ℓ_2}	R_{ℓ_3}
e						
P_{120}						
P_{240}						
R_{ℓ_1}						
R_{ℓ_2}						
R_{ℓ_3}						

9. Does the table illustrate an operational system?

10. Does the system have an identity element? Name it.

11. Does the system have the inverse property? Name the inverse of each element if one exists.

12. Is the system commutative? Why?

13. Is the system associative? Check these two examples.

a) $(R_{\ell_3} \circ P_{120}) \circ R_{\ell_1} = R_{\ell_3} \circ (P_{120} \circ R_{\ell_1})$

b) $(P_{120} \circ R_{\ell_2}) \circ P_{240} = P_{120} \circ (R_{\ell_2} \circ P_{240})$

14. Is the system a group? Is it a commutative group.

15. Complete the table for the isometries of a square.

$\circ$	e	P_{90}	P_{180}	P_{270}	R_{ℓ_1}	R_{ℓ_2}	R_{ℓ_3}	R_{ℓ_4}
e								
P_{90}								
P_{180}								
P_{270}								
R_{ℓ_1}								
R_{ℓ_2}								
R_{ℓ_3}								
R_{ℓ_4}								

16. Does the table illustrate an operational system?

17. Does the system have an identity element? Name it.

18. Does the system have the inverse property? Name the inverse of each element if one exists.

19. Is the system commutative? Why?

20. Is the system associative? Check these examples.

a) $(R_{\ell_1} \circ R_{\ell_2}) \circ R_{\ell_3} = R_{\ell_1} \circ (R_{\ell_2} \circ R_{\ell_3})$

b) $(P_{90} \circ R_{\ell_3}) \circ P_{90} = P_{90} \circ (R_{\ell_3} \circ P_{90})$

21. Is the system a group? Is it a commutative group?

Section 10. Dilation

There is one type of plane transformation that does not preserve the distance between points. A dilation is not an isometry, but it is an equally important plane transformation. In a dilation, both coordinates are multiplied by the same whole number. The rule of assignment is therefore

$$(x,y) \rightarrow (a \cdot x,\ a \cdot y)$$

A dilation is denoted D_a; a is called the factor of dilation. For the present a will represent any whole number. Later the set of factors of dilation will be extended to include negative integers and fractions.

To see how this type of transformation affects a figure, find the image of ΔABC under D_2 (a dilation by a factor of 2). The points A, B, and C and their images are listed below.

$$A(-2,-1) \rightarrow A'(2\cdot-2, 2\cdot-1) = A'(-4,-2)$$
$$B(3,-1) \rightarrow B'(2\cdot3, 2\cdot-1) = B'(6,-2)$$
$$C(-2,3) \rightarrow C'(2\cdot-2, 2\cdot3) = C'(-4,6)$$

The graph in Fig. 1 show both ΔABC and its image $\Delta A'B'C'$.

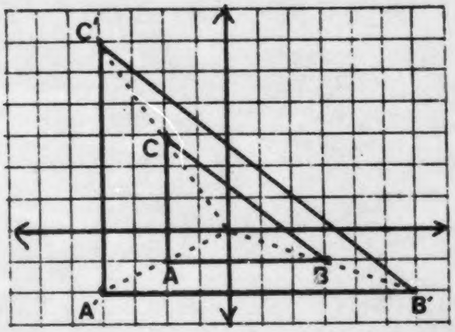


Fig. 1

Several important facts about dilations can be seen from the figure. The distance from A to B is 5 units while the distance from A' to B' is 10 units. Likewise the distance from A to C is 4 units and that from A' to C' is 8 units. Therefore, you may conclude that dilating two points by a factor of 2, multiplies the distance between them by the same factor. Notice, also, that the corresponding sides of the two triangles are parallel. This is true in every dilation.

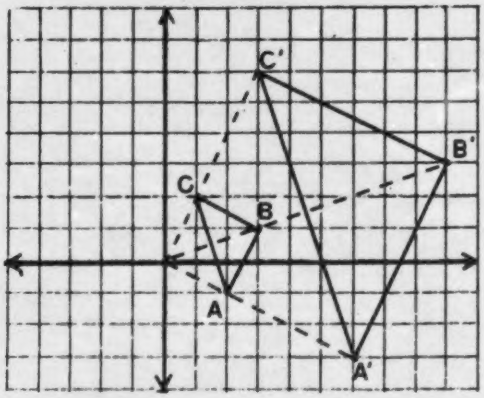
Now consider the dashed lines coming out from the origin. Both A and A' lie on the same segment as do B and B', and C and C'. In addition, the distance from (0,0) to A' is exactly two times the distance from the origin to A. The same is true for the other points. You may wish to check this with your ruler.

The last and perhaps the most important conclusion to be drawn from the figure above concerns the overall shape of the triangles. The shape ΔA'B'C' is basically the same as the shape of ΔABC, only larger. That is to say, the corresponding angles in both triangles have the same measure; for example, ∠A and ∠A' are both right angles. The two other pairs of angles can be shown to be of equal measure by tracing one, for example ∠C and placing it over ∠C'. The two will match perfectly, as well ∠B and ∠B'. In all dilations the angles measures remain the same while the lengths of the sides change according to the factor of dilation.

In some cases, the origin, the center of the dilation is not in the interior of the triangle. Example 1 shows a dilation by a factor of 3 where the origin is outside the triangle.

Ex. 1

A(2,-1) → A'(6,-3)
B(3,1) → B'(9,3)
C(1,2) → C'(3,6)



Measure the distances between each pair of points in both triangles. Also measure the distances from the origin to the vertices of ΔABC and to the vertices of $\Delta A'B'C'$. In all cases you should find that the distances for $\Delta A'B'C'$ are exactly 3 times those for ΔABC. Also corresponding sides of the triangles are parallel and the corresponding angles have the same measure just as in the previous example.

EXERCISES

Find the image of each triangle and the dilation given. List the vertices and their images. Also graph both triangles.

1. ΔABC under D_3 A(0,0),; B(2,0); C(1,3)

2. ΔDEF under D_2 D(-3,2); E(-2,-2); F(1,-1)

3. ΔRST under D_2 R(-4,2); S(-3,-1); T(2,-3)

4. ΔXYZ under D_1 X(2,5); Y(-3,4); Z(1,-2)

5. Find the image of ΔKLM under D_2 where K(-1,2); L(0,-1); M(2,1). Then find the image of $\Delta K'L'M'$ under D_3.

6. Can the two dilations in problem 5 be combined into one dilation? What is $D_3 \circ D_2$?

7. Can any two dilations be combined into one dilation?

8. Is $(D_a, \circ)$ an operational system?

9. Complete this statement, $D_b \circ D_a =$

10. Is there an identity dilation? If so name it; if not, explain why not.

11. What is the inverse of D_2? of D_3?

12. What dilation has an inverse?

13. Is $(D_a, \circ)$ associative? Simplify these examples.

    a) $D_2 \circ (D_3 \circ D_4) = (D_2 \circ D_3) \circ D_4$.

    b) $(D_4 \circ D_4) \circ D_2 = D_4 \circ (D_4 \circ D_2)$

14. Is $(D_a, \circ)$ a group?

15. Is $(D_a, \circ)$ commutative? Why?

16. Find the image of ΔXYZ under D_0 where X(2,4); Y(5,1); Z(-2,-4).

17. What is the image of any figure under D_0?

18. Perform the transformation D_2 on ΔEFG where E(2,3), F(1,0), and G(0,2). Graph both triangles. Next perform the transformation P_{180} on $\Delta E'F'G'$ Label the final image $\Delta E''F''G''$.

19. Make a list of points and images from problem 18. What single factor of dilation maps ΔEFG to $\Delta E''F''G''$? All negative dilations are equivalent to the composition of a dilation where $a > 0$ and a rotation of 180 degrees.

20. Does $P_{180} \circ D_2 = D_2 \circ P_{180}$?

21. What single dilation is equivalent to $D_3 \circ P_{180}$?

REVIEW CHAPTER 5 TRANSFORMATIONAL GEOMETRY

1. Define these terms:
   plane transformation
   isometry
   translation
   line reflection
   rotation
   dilation

2. Name four different lines of reflection and give the rule of assignment for each.

3. Name four different rotations and give the rule of assignment for each.

4. Give the rule of assignment for any translation.

5. Give the rule of assignment for any dilation.

6. Find the image of the point (4,-3) under each of the following transformations:
   a) R_x, b) R_y, c) R_{D1}, d) R_{D2}, e) P_0,
   f) P_{90}, g) P_{180}, h) P_{270}, i) $T_{1,-5}$, j) D_3.

Answers

1. See text.
2. $R_x: (x,y) \to (x,-y)$; $R_y: (x,y) \to (-x,y)$
   $R_{D1}: (x,y) \to (y,x)$; $R_{D2}: (x,y) \to (-y,-x)$

3. $P_0: (x,y) \to (x,y)$; $P_{90}: (x,y) \to (-y,x)$
   $P_{180}: (x,y) \to (-x,-y)$; $P_{270}: (x,y) \to (y,-x)$

4. $T_{a,b}: (x,y) \to (x+a,y+b)$

5. $D_a: (x,y) \to (a \cdot x, a \cdot y)$

6. a) (4,3), b) (-4,-3), c) (-3,4), d) (3,-4), e) (4,-3), f) (3,4), g) (-4,3), h) (-3,-4), i) (5,-8), j) (12,-9)